Chapter 3

MATRICES

EXERCISE 3.1 (Page 126)

1. Let

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

Find: (i) $A + B$

Sol. $$A + B = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & -3+3 & -5+5 \\ -1+1 & 4-3 & 5-5 \\ 1-1 & -3+3 & -4+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1.(ii) $A - B$

Sol. $$A - B = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & -3-3 & -5-5 \\ -1-1 & 4+3 & 5+5 \\ 1+1 & -3-3 & -4-3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -10 \\ -2 & 7 & 10 \\ 2 & -6 & -7 \end{bmatrix}$$

1.(iii) $2A+3B$

Sol. $$2A+3B = 2\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} + 3\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 & -10 \\ -2 & 8 & 10 \\ 2 & -6 & -8 \end{bmatrix} + \begin{bmatrix} -3 & 9 & 15 \\ 3 & -9 & -15 \\ -3 & 9 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+9 & -10+15 \\ -2+3 & 8-9 & 10-15 \\ 2-3 & -6+9 & -8+9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 1 & -1 & -5 \\ -1 & 3 & 1 \end{bmatrix}$$

1.(iv) $3A-5B$

Sol. $$3A-5B = 3\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} - 5\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -9 & -15 \\ -3 & 12 & 15 \\ 3 & -9 & -12 \end{bmatrix} + \begin{bmatrix} 5 & -15 & -25 \\ -5 & 15 & 25 \\ 5 & -15 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 6+5 & -9-15 & -15-25 \\ -3-5 & 12+15 & 15+25 \\ 3+5 & -9-15 & -12-15 \end{bmatrix} = \begin{bmatrix} 11 & -24 & -40 \\ -8 & 27 & 40 \\ 8 & -24 & -27 \end{bmatrix}$$

1.(v) AB

Sol.

$$AB = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times(-1)-3\times1-5\times(-1) & 2\times3-3\times(-3)-5\times3 & 2\times5-3\times(-5)-5\times3 \\ -1\times(-1)+4\times1+5\times(-1) & -1\times3+4\times(-3)+5\times3 & -1\times5+4\times(-5)+5\times3 \\ 1\times(-1)-3\times1-4\times(-1) & 1\times3-3\times(-3)-4\times3 & 1\times5-3\times(-5)-4\times3 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+5 & 6+8-15 & 10+15-15 \\ 1+4-5 & -3-12+15 & -5-20+15 \\ -1-3+5 & 3+9-12 & 5+15-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & -10 \\ 0 & 0 & 8 \end{bmatrix}$$

1.(vi) BA

Sol.

$$BA = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times2+3\times(-1)+5\times1 & -1\times(-3)+3\times4+5\times(-3) & -1\times(-5)+3\times5+5\times(-4) \\ 1\times2-3\times(-1)-5\times1 & 1\times(-3)-3\times4-5\times(-3) & 1\times(-5)-3\times5-5\times(-4) \\ -1\times2+3\times(-1)+3\times1 & -1\times(-3)+3\times4+3\times(-3) & -1\times(-5)+3\times5+3\times(-4) \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+5 & 3+12-15 & 5+15-20 \\ 2+3-5 & -3-12+15 & -5-15+20 \\ -2-3+3 & 3+12-9 & 5+15-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 6 & 8 \end{bmatrix}$$

2. (i) Evaluate $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$

Sol. $$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2-2 & -4+10+2 \\ 8+0-4 & -16+0+4 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & -12 \end{bmatrix}$$

2.(ii) $[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Sol. $$[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x\ y\ z]\begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$$

$$= [\,x(ax+hy+gz)+y\,(hx+by+fz\,)+z(gx+fy+cz)]$$

$$= [ax^2+hxy+gzx+hxy+by^2+fyz+gzx+fyz+cz^2]$$

$$= [ax^2+by^2+cz^3+2hxy+2fyz+2gzx]$$

2.(iii) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2$

Sol. $$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 2-6 \\ 4-12 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$$

2.(iv) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3$

Sol.
$$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}, \text{ From 2}(iii) \text{ above}$$
$$= \begin{bmatrix} 9-16 & 18+12 \\ -8+68 & -16-51 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$$

2.(v) $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2$

Sol.
$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-0-0 & 1-1-0 & 1-0-1 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.(vi) $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3$

Sol.
$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ by 2(v) above}$$
$$= \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Prove that the product of matrices

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is the zero matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Sol.
$$AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$
$$= \cos(\theta-\phi)\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix} = 0, \text{ if and only if } \cos(\theta-\phi) = 0.$$

Hence $AB = 0, \quad \Leftrightarrow \quad \cos(\theta-\phi) = 0$

or $AB = 0, \quad \Leftrightarrow \quad \theta - \phi = (2n+1)\frac{\pi}{2}$, where n is an integer.

4. The direction cosines of two lines are $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3. Prove that the product

$$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_1\mu_2 & \mu_2^2 & \mu_2\mu_3 \\ \mu_1\mu_3 & \mu_2\mu_3 & \mu_3^2 \end{bmatrix}$$

is zero if and only if the lines are perpendicular to each other.

Sol.
$$AB = \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_1\mu_2 & \mu_2^2 & \mu_2\mu_3 \\ \mu_1\mu_3 & \mu_2\mu_3 & \mu_3^2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1^2\mu_1^2 + \lambda_1\lambda_2\mu_1\mu_2 + \lambda_1\lambda_3\mu_1\mu_3 & \lambda_1^2\mu_1\mu_2 + \lambda_1\lambda_2\mu_2^2 + \lambda_1\lambda_3\mu_2\mu_3 & \lambda_1^2\mu_1\mu_3 + \lambda_1\lambda_2\mu_2\mu_3 + \lambda_1\lambda_3\mu_3^2 \\ \lambda_1\lambda_2\mu_1^2 + \lambda_2^2\mu_1\mu_2 + \lambda_2\lambda_3\mu_1\mu_3 & \lambda_1\lambda_2\mu_1\mu_2 + \lambda_2^2\mu_2^2 + \lambda_2\lambda_3\mu_2\mu_3 & \lambda_1\lambda_2\mu_1\mu_3 + \lambda_2^2\mu_2\mu_3 + \lambda_2\lambda_3\mu_3^2 \\ \lambda_1\lambda_3\mu_1^2 + \lambda_2\lambda_3\mu_1\mu_2 + \lambda_3^2\mu_1\mu_3 & \lambda_1\lambda_3\mu_1\mu_2 + \lambda_2\lambda_3\mu_2^2 + \lambda_3^2\mu_2\mu_3 & \lambda_1\lambda_3\mu_1\mu_3 + \lambda_2\lambda_3\mu_2\mu_3 + \lambda_3^2\mu_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_2\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_3\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_3\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_3\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \end{bmatrix}$$

$$= (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)\begin{bmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{bmatrix}$$

Hence $AB = 0$ if and only if $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$.

i.e., if and only if the two lines are perpendicular.

5. Show that, in general, for any two matrices A and B which are conformable for addition and multiplication,

$$(A+B)^2 \neq A^2 + 2AB + B^2$$

and $$A^2 - B^2 \neq (A-B)(A+B)$$

Under what conditions equality holds in each case?

Sol. $(A+B)^2 = (A+B(A+B) = A^2 + AB + BA + B^2$

Since $AB \neq BA$ in general, so we must have

$$(A+B)^2 \neq A^2 + 2AB + B^2$$

$$(A-B)(A+B) = A^2 + AB - BA - B^2 \neq A^2 - B^2.$$

since $AB \neq BA$ is general.

Equality holds in each case if $AB = BA$.

6. If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{ show that}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and}$$

$$A^3 - 3A^2 - 7A - 3I = 0.$$

Sol. $A^2 - 4A - 5I$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+2 & 2+2+2 & 1+4+1 \\ 2+2+4 & 4+1+4 & 2+2+2 \\ 2+4+2 & 4+2+2 & 2+4+1 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 4 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} + \begin{bmatrix} -9 & -8 & -4 \\ -8 & -9 & -8 \\ -8 & -8 & -9 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^3 - 3A^2 - 7A - 3I = A \cdot A^2 - 3A^2 - 7A - 3I$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} - 3\begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7+16+8 & 6+18+8 & 6+12+7 \\ 14+8+16 & 12+9+16 & 12+6+14 \\ 14+16+8 & 12+18+8 & 12+12+7 \end{bmatrix} - \begin{bmatrix} 21 & 18 & 18 \\ 24 & 27 & 18 \\ 24 & 24 & 21 \end{bmatrix}$$

$$- \begin{bmatrix} 7 & 14 & 7 \\ 14 & 7 & 14 \\ 14 & 14 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 32 & 25 \\ 38 & 37 & 32 \\ 38 & 38 & 31 \end{bmatrix} + \begin{bmatrix} -21-7-3 & -18-14 & -18-7 \\ -24-14 & -27-7-3 & -18-14 \\ -24-14 & -24-14 & -21-7-3 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 32 & 25 \\ 38 & 37 & 32 \\ 38 & 38 & 31 \end{bmatrix} + \begin{bmatrix} -31 & -32 & -25 \\ -38 & -37 & -32 \\ -38 & -38 & -31 \end{bmatrix} = 0.$$

7. Show that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \text{ is periodic having period 2.}$$

Sol. $$A^2 = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-12 & -2-4+0 & -6-18+18 \\ -3-6+18 & 6+4+0 & 18+18-27 \\ 2+0-6 & -4+0+0 & -12+0+9 \end{bmatrix} = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+18-12 & 10-12+0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-8+0 & 24-36+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = A$$

Thus A is periodic having period 2.

8. Show that

$$\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$ is nilpotent. What is its nilpotency index?

Sol. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Hence A is nilpotent of index 2.

9. Show that

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$ is involutory.

Sol. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is involutory.

10. Show that every square matrix A with entries from R, can be written as a symmetric matrix $B = \frac{1}{2}(A + A^T)$ and a skew symmetric matrix $D = \frac{1}{2}(A - A^T)$.

Sol. Since we can write every square matrix A as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = B + D,$$

where $B = \frac{1}{2}(A + A^T), D = \frac{1}{2}(A - A^T)$.

We have to prove that B is symmetric and D is skew symmetric

Now $B^T = \left[\frac{1}{2}(A + A^T)\right]^T, \quad D^T = \left[\frac{1}{2}(A - A^T)\right]^T$

or $B^T = \frac{1}{2}\left[A^T + (A^T)^T\right], \quad D^T = \frac{1}{2}\left[A^T - (A^T)^T\right]$

or $B^T = \frac{1}{2}\left[A^T + A\right], \quad D^T = \frac{1}{2}\left[A^T - A\right]$, as $(A^T)^T = A$

or $B^T = \frac{1}{2}\left(A + A^T\right) = B, \quad D^T = -\frac{1}{2}\left(A - A^T\right) = -D.$

This shows that the matrix B is symmetric and D is skew symmetric.

11. Show that every square matrix A with entries from C, can be written as a Hermitian matrix $B = \frac{1}{2}(A + (\bar{A})^T)$, and as a skew Hermitian matrix $D = \frac{1}{2}(A + (-\bar{A})^T)$.

Sol. Since we can write every square matrix A is

$$A = \frac{1}{2}(A+(\bar{A})^T)+\frac{1}{2}(A-(\bar{A})^T) = B+D$$

where $B = \frac{1}{2}(A+(\bar{A})^T),\ D = \frac{1}{2}(A-(\bar{A})^T)$

We prove that B is Hermitian and D is skew Hermitian.

Now $\bar{B} = \frac{1}{2}\overline{(A+(\bar{A})^T)}$, $\quad \bar{D} = \frac{1}{2}\overline{(A-(\bar{A})^T)}$

or $\bar{B} = \frac{1}{2}[\bar{A}+\overline{(\bar{A})}^T]$, $\quad \bar{D} = \frac{1}{2}[\bar{A}-\overline{(\bar{A})}^T]$

or $\bar{B} = \frac{1}{2}[\bar{A}+A^T]$, $\quad \bar{D} = \frac{1}{2}[\bar{A}-A^T]$

$\Rightarrow (\bar{B})^T = \frac{1}{2}[\bar{A}+A^T]^T$, $\quad (\bar{D})^T = \frac{1}{2}[\bar{A}-A^T]^T$

or $(\bar{B})^T = \frac{1}{2}[(\bar{A})^T+(A^T)^T]$, $(\bar{D})^T = \frac{1}{2}[(\bar{A})^T-(A^T)^T]$

or $(\bar{B})^T = \frac{1}{2}[(\bar{A})^T+A]$, $\quad (\bar{D})^T = \frac{1}{2}[(\bar{A})^T-A]$

or $(\bar{B})^T = \frac{1}{2}(A+(\bar{A})^T) = B$, $(\bar{D})^T = -\frac{1}{2}(A-(\bar{A})^T) = -D$.

This shows that the matrix B is Hermitian and the matrix D is skew Hermitian.

12. If A and B are symmetric matrices, then prove that AB is symmetric if and only if A and B commute.

Sol. Since A and B are symmetric, we have

$$A^T = A, \quad B^T = B$$

Suppose AB is symmetric, then

$$(AB)^T = AB$$
$$\Rightarrow \quad B^TA^T = AB$$
$$\Rightarrow \quad BA = AB$$

showing that A and B commute.

Conversely, suppose $AB = BA$, then

$$(AB)^T = (BA)^T$$
$$\Rightarrow \quad (AB)^T = A^TB^T = AB$$

showing that AB is symmetric.

13. If A is an $m \times m$ symmetric (skew symmetric) matrix and P is an $m \times n$ matrix, then prove that $B = P^T AP$ is symmetric (skew symmetric).

Sol. Since P is an $m \times n$ matrix, P^T is an $n \times m$ matrix. So $B = P^TAP$ is an $n \times n$ matrix.

When A is symmetric, i.e., $A^T = A$, then

$$B^T = (P^TAP)^T = P^TA^T(P^T)^T = P^TAP = B,$$

showing that B is symmetric.

When A is skew symmetric, i.e., $A^T = -A$, then

$$B^T = (P^TAP)^T = P^TA^T(P^T)^T = P^T(-A)P = -P^TAP = -B,$$

showing that B is skew symmetric.

14. Show that AA^T and A^TA are symmetric for any square matrix A.

Sol. Using $(AB)^T = B^TA^T$, we have

$$(AA^T)^T = (A^T)^TA^T = AA^T.$$

Hence AA^T is symmetric.

Again, $(A^TA)^T = A^T(A^T)^T = A^TA$.

Therefore, A^TA is symmetric.

15. If A is a square matrix over C, then show that $A+(\bar{A})^T$, $A(\bar{A})^T$ and $(\bar{A})^TA$ are all Hermitian.

Sol. Let $B = A+(\bar{A})^T$, then

$$(\bar{B})^T = \left[\overline{A+(\bar{A})^T}\right]^T = [\bar{A}+\overline{(\bar{A})^T}]^T$$
$$= (\bar{A}+A^T)^T = (\bar{A})^T+(A^T)^T = (\bar{A})^T+A$$

showing that D is Hermitian.

Finally, let $E = (\bar{A})^TA$, then

$$\bar{E}^T = [\overline{(\bar{A}^TA)}]^T = [(\bar{A})^T\bar{A}]^T = [(\bar{\bar{A}})^T\bar{A}]^T = (A^T\bar{A})^T$$
$$= [(\bar{A})^T(A^T)^T] = (\bar{A})^TA = E,$$

showing that E is Hermitian.

16. Show that every square matrix over C can be expressed in a unique way as $P+iQ$, where P and Q are Hermitian.

Sol. Since we can write every square matrix A as

$$A = \frac{1}{2}(A+(\bar{A})^T)+\frac{1}{2}(A-(\bar{A})^T) = P+iQ$$

where $P = \frac{1}{2}(A+(\bar{A})^T)$, $Q = \frac{1}{2i}(A-(\bar{A})^T)$.

We have to prove that P and Q are Hermitian.

Now $\bar{P} = \frac{1}{2}\overline{(A+(\bar{A})^T)}$, $\quad \bar{Q} = \frac{1}{-2i}\overline{(A-(\bar{A})^T)}$

or $\bar{P} = \frac{1}{2}[\bar{A}+\overline{(\bar{A})}^T]$, $\quad \bar{Q} = \frac{-1}{2i}[\bar{A}-\overline{(\bar{A})}^T]$

or $\bar{P} = \frac{1}{2}[\bar{A}+A^T]$, $\quad \bar{Q} = \frac{-1}{2i}[\bar{A}-A^T]$

$\Rightarrow (\bar{P})^T = \frac{1}{2}[\bar{A}+A^T]^T$, $\quad (\bar{Q})^T = \frac{-1}{2i}[\bar{A}-A^T]^T$

or $(\bar{P})^T = \frac{1}{2}[(\bar{A})^T+(A^T)^T]$, $\quad (\bar{Q})^T = \frac{-1}{2i}[(\bar{A})^T-(A^T)^T]$

or $(\bar{P})^T = \frac{1}{2}[(\bar{A})^T+A]$, $\quad (\bar{Q})^T = \frac{-1}{2i}[(\bar{A})^T-A]$

or $(\bar{P})^T = \frac{1}{2}(A+(\bar{A})^T)=P$, $\quad (\bar{Q})^T = \frac{1}{2i}(A-(\bar{A})^T)=-Q$

This shows that P and Q are Hermitian.

Now we prove that $A = P + iQ$ is unique.

On the contrary, suppose that there exist Hermitian matrices S and T such that $A = S + iT$ (1)

$\Rightarrow (\bar{A})^T = [\overline{S+iT}]^T$

$\Rightarrow (\bar{A})^T = [\overline{S+iT}]^T = [\bar{S}+\overline{iT}]^T = [\bar{S}-i\bar{T}]^T = (\bar{S})^T - i(\bar{T})^T$ (2)

$= S - iT$

as S and T are Hermitian.

Adding equations (1) and (2), we get

$$A+(\bar{A})^T = 2S \Rightarrow S = \frac{1}{2}(A+(\bar{A})^T) = P$$

Subtracting equation (2) from equation (1), we have

$$A-(\bar{A})^T = 2iT \Rightarrow T = \frac{1}{2i}(A-(\bar{A})^T) = Q$$

Thus the expression $A = P + iQ$ is unique.

17. Show that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew symmetric.

Sol. Let P be a Hermitian matrix, then P is a square matrix over C and

$(\bar{P})^T = P$.

Let $P = [x_{ij}]_{n\times n}$, where $x_{ij} \in C$, such that

$x_{ij} = a_{ij} + ib_{ij}$, where $a_{ij}, b_{ij} \in R$, $i = \sqrt{-1}$. Then

$P = [a_{ij}+ib_{ij}] = [a_{ij}] + i[b_{ij}] = A + iB$, where

$A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ real matrices, and so

$\bar{A} = A$, $\bar{B} = B$

Since P is Hermitian, therefore $P = (\bar{P})^T$

or $A + iB = (\overline{A+iB})^T = (\bar{A}+\overline{iB})^T$

$= (\bar{A}+\bar{i}\,\bar{B})^T = (A-iB)^T$, as $\bar{A} = A$, $\bar{B} = B$

$= A^T - iB^T$

Equating real and imaginary parts, we have

$A^T = A$, $B^T = -B$

showing that A is real and symmetric and B is real and skew symmetric.

18. Show that, for any matrices A and B over C and $k \in C$,

(i) $\bar{\bar{A}} = A$

Sol. Let $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$

Thus $\bar{\bar{A}} = [\bar{\bar{a}}_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$

18.(ii) $\overline{kA} = \bar{k}\bar{A}$

Sol. Let $A = [a_{ij}]_{m\times n}$, then

$kA = [ka_{ij}]_{m\times n}$

Thus $\overline{kA} = [\bar{k}\bar{a}_{ij}]_{m\times n} = \bar{k}[\bar{a}_{ij}]_{m\times n} = \bar{k}\bar{A}$

18.(iii) $\overline{A+B} = \bar{A}+\bar{B}$

Sol. Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, then $A+B = [a_{ij}+b_{ij}]_{m\times n}$

Thus $(\overline{A+B}) = [\overline{a_{ij}+b_{ij}}]_{m\times n} = [\bar{a}_{ij}+\bar{b}_{ij}]_{m\times n}$

$= [\bar{a}_{ij}]_{m\times n} + [\bar{b}_{ij}]_{m\times n} = \bar{A}+\bar{B}$

18.(iv) $\overline{AB} = \bar{A}\bar{B}$

Sol. Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, then

$$AB = \left[\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right]_{m\times p}$$

$$(\overline{A}\,\overline{B}) = \overline{\left[\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right]}_{m\times p}$$

$$= \left[\sum_{\lambda=1}^{n} \overline{a_{i\lambda} b_{\lambda j}}\right]_{m\times p}, \text{ Distributivity of conjugation over sum.}$$

$$= \left[\sum_{\lambda=1}^{n} \overline{a}_{i\lambda} \overline{b}_{\lambda j}\right]_{m\times p} = \overline{A} + \overline{B}$$

18.(v) $(\overline{A})^T = \overline{(A)^T}$.

Sol. Let $A = [a_{ij}]_{m\times n}$ then $\overline{A} = [\overline{a}_{ij}]_{m\times n}$

$= [\alpha_{ij}]_{m\times n}$ where $\alpha_{ij} = \overline{a}_{ij}$

$\overline{(A)}^T = [x_{ij}]_{n\times m}$ where $x_{ij} = \overline{\alpha}_{ij}$

Now $\overline{a}_{ij}$ is the entry lying in the ith row and jth column of $\overline{(A)}^T$.

Hence $\overline{(A)}^T = [x_{ij}]_{n\times m} = \overline{(A)}$.

19. If A is a matrix over R and $AA^T = 0$, show that $A = 0$.

Sol. Let $A = [a_{ij}]_{m\times n}$ then

$A^T = [b_{ij}]_{n\times m}$, where $b_{ij} = a_{ji}$ (1)

Now $0 = AA^T = \left[\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right]_{m\times m}$

$$\Rightarrow \sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j} = 0, \text{ for all } i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, m.$$

Replacing i by λ in equation (1), we get $b_{\lambda j} = a_{j\lambda}$, so the above equation becomes

$$\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j} = 0, \text{ for all } i = j = 1, 2, 3, \cdots, m$$

or $$\sum_{\lambda=1}^{n} (a_{i\lambda})^2 = 0, \text{ for all } i = j = 1, 2, 3, \cdots, m$$

$$\Rightarrow (a_{i\lambda})^2 = 0, \text{ for all } i = j = 1, 2, 3, \cdots, m;\ \lambda = 1, 2, 3, \cdots, n$$

$$\Rightarrow a_{i\lambda} = 0, \text{ for all } i = j = 1, 2, 3, \cdots, m;\ \lambda = 1, 2, 3, \cdots, n$$

Hence every entries of the matrix A is zero, so $A = 0$.

20. If A is a matrix over C and $A(\overline{A})^T = 0$, then show that $\overline{A} = 0 = A$.

Sol. Let $A = [a_{ij}]_{m\times n}$, for $a_{ij} \in C$, then $\overline{A} = [\overline{a}_{ij}]_{m\times n}$

and $(\overline{A})^T = [b_{ij}]_{n\times m}$, where $b_{ij} = \overline{a}_{ji}$ (1)

Now $0 = A(\overline{A})^T = \left[\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right]_{m\times m}$

$$\Rightarrow \sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j} = 0, \text{ for all } i = 1, 2, 3, \cdots, m; j = 1, 2, 3, \cdots, m$$

Replacing i by λ in equation (1), we get $b_{\lambda j} = \overline{a}_{j\lambda}$, so the above equation becomes

$$\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j} = 0, \text{ for all } i = 1, 2, 3, \cdots, m; j = 1, 2, 3, \cdots, m$$

$$\Rightarrow \sum_{\lambda=1}^{n} a_{i\lambda} \overline{a}_{i\lambda} = 0, \text{ for all } i = j = 1, 2, 3, \cdots, m$$

or $$\sum_{\lambda=1}^{n} |a_{i\lambda}|^2 = 0, \text{ for all } i = 1, 2, 3, \cdots, m$$

$$\Rightarrow |a_{i\lambda}|^2 = 0, \text{ for all } i = 1, 2, 3, \cdots, m;\ \lambda = 1, 2, 3, \cdots, n$$

$$\Rightarrow a_{i\lambda} = 0, \text{ for all } i = 1, 2, 3, \cdots, m;\ \lambda = 1, 2, 3, \cdots, n$$

and so $\overline{a}_{i\lambda} = 0$, for all $i = 1, 2, 3, \cdots, m;\ \lambda = 1, 2, 3, \cdots, n$

Hence every entry of the entire A as well as of $\overline{A}$ is zero, so $A = 0$ and $\overline{A} = 0$.

21. Show that for any matrix A, $(A^p)^T = (A^T)^p$, where p is a positive integer.

Sol. $A^p = A \cdot A \cdot \cdots \cdot A$ p times

$\Rightarrow (A^p)^T = (A \cdot A \cdot \cdots \cdot A)^T$

$= A^T \cdots A^T \cdot A^T$ p times

$= (A^T)^p$

22. Let $x = [x_1\ \ x_2\ \ x_3]$, $y = [y_1\ \ y_2\ \ y_3]$ be two 1×3 matrices. Define a product $x \times y$ (called a **vector product** of x and y), as follows:

$x \times y = [x_2y_3 - y_2x_3\ \ \ x_3y_1 - y_3x_1\ \ \ x_1y_2 - y_1x_2]$

Show that

(i) $y \times x = -(x \times y)$ (ii) $x \times x = 0$ for all x.

(iii) $(x \times y) \times z \neq x \times (y \times z)$, where $z = [z_1\ \ z_2\ \ z_3]$

Sol. (i) $x = [x_1 \;\; x_2 \;\; x_3]$, $y = [y_1 \;\; y_2 \;\; y_3]$

$$x \times y = [x_2y_3 - x_3y_2 \quad x_3y_1 - x_1y_3 \quad x_1y_2 - x_2y_1]$$

$$y \times x = [y_2x_3 - y_3x_2 \quad y_3x_1 - y_1x_3 \quad y_1x_2 - y_2x_1]$$

$$= [-(x_2y_3 - x_3y_2) \quad -(x_3y_1 - x_1y_3) \quad -(x_1y_2 - x_2y_1)] = -(x \times y)$$

(ii) $x \times x = [x_2x_3 - x_3x_2 \quad x_3x_1 - x_1x_3 \quad x_1x_2 - x_2x_1] = [0 \;\; 0 \;\; 0] = 0$

(iii) $(x \times y) \times z = [x_2y_3 - x_3y_2 \quad x_3y_1 - x_1y_3 \quad x_1y_2 - x_2y_1] \times [z_1 \;\; z_2 \;\; z_3]$

$$= [(x_3y_1 - x_1y_3)z_3 - (x_1y_2 - x_2y_1)z_2 \quad (x_1y_2 - x_2y_1)z_1 - (x_2y_3 - x_3y_2)z_3 \quad (x_2y_3 - x_3y_2)z_2 - (x_3y_1 - x_1y_3)z_1] \quad (1)$$

$$x \times (y \times z) = [x_1 \;\; x_2 \;\; x_3] \times [y_2z_3 - y_3z_2 \quad y_3z_1 - y_1z_3 \quad y_1z_2 - y_2z_1]$$

$$= [x_2(y_1z_2 - y_2z_1) - x_3(y_3(y_3z_1 - y_1z_3) \quad x_3(y_2z_3 - y_3z_2) - x_1(y_1z_2 - y_2z_1) \quad x_1(y_3z_1 - y_1z_3) - x_2(y_2z_3 - y_3z_2)] \quad (2)$$

From (1) and (2) it is clear that $(x \times y) \times z \neq x \times (y \times z)$.

23. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Define **inner product** of A and B by:

$$A \cdot B = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

Show that:

(i) $A \cdot B = 0$ but neither $A = 0$ nor $B = 0$

(ii) $A \cdot B = B \cdot A$

(iii) $(\alpha A + \beta B) \cdot C = \alpha A \cdot C + \beta B \cdot C$, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

(iv) $A \cdot A \geq 0$ and $A \cdot A = 0 \Leftrightarrow A = 0$.

Sol. (i) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

then $A \cdot B = 2 - 2 - 12 + 12 = 0$, but neither $A = 0$ nor $B = 0$.

23.(ii)
$$B \cdot A = b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22}$$
$$= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} = A \cdot B$$

23.(iii)
$$\alpha A + \beta B = \begin{bmatrix} \alpha a_{11} + \beta b_{11} & \alpha a_{12} + \beta b_{12} \\ \alpha a_{21} + \beta b_{21} & \alpha a_{22} + \beta b_{22} \end{bmatrix}$$

$$(\alpha A + \beta B) \cdot C = (aa_{11} + bb_{11})\,c_{11} + (aa_{12} + bb_{12})\,c13\,(aa_{21} + bb_{21})\,c_{21} + (aa_{22} + bb_{22})\,c_{22}$$
$$= aa_{11}c_{11} + aa_{12}c_{12} + (aa_{22} + bb_{22})c_{22}$$
$$= aa_{11}c_{11} + aa_{12}c_{12} + aa_{21}c_{21} + aa + aa_{22}c_{22} + bb_{11}c_{11} + bb_{12}c_{12} + bb_{21}c_{21} + bb$$
$$= aa \cdot C + bb \cdot C$$

23.(iv)
$$A \cdot A = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$$

and $A \cdot A = 0 \Leftrightarrow$

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$$
$$\Leftrightarrow a_{11} = a_{12} = a_{21} = a_{22} = 0$$
$$\Leftrightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

24. Let

$$A = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

and

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 3 & 6 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Compute AB using the indicated partitionings.

Sol. Let $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $0_{12} = [0 \;\; 0]$, $S = [2]$. Then

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} P & Q \\ 0_{12} & S \end{bmatrix}$$

Suppose $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $0_{13} = [0 \;\; 0 \;\; 0]$, $I = [1]$. Then

$$B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 3 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C & D \\ 0_{13} & I \end{bmatrix}$$

Now
$$AB = \begin{bmatrix} P & Q \\ 0_{12} & S \end{bmatrix} \begin{bmatrix} C & D \\ 0_{13} & I \end{bmatrix}$$
$$= \begin{bmatrix} PC + Q0_{13} & PD + QI \\ 0_{12}C + S0_{13} & 0_{12}D + SI \end{bmatrix}$$

$$PC + Q0_{13} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \;\; 0 \;\; 0]$$
$$= \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix}$$

$$PD + QI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1] = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$0_{12}C + S0_{13} = [0\ 0]\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + [2][0\ 0\ 0]$$

$$= [0\ 0\ 0] + [0\ 0\ 0] = [0\ 0\ 0]$$

$$0_{12}D + SI = [0\ 0]\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [2][1] = [0] + [2] = [2]$$

Hence $AB = \begin{bmatrix} PC + Q0_{13} & PD + QI \\ 0_{12}C + S0_{13} & 0_{12}D + SI \end{bmatrix}$

$$= \left[\begin{array}{ccc|c} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ \hline 0 & 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

25. Let

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } B = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find AB using the indicated partitionings.

Sol. Let $P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $0_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $0_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also $S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

Thus $A = \begin{bmatrix} P & 0_{23} \\ 0_{32} & I \end{bmatrix}$, $B = \begin{bmatrix} 0_{23} & Q \\ S & 0_{32} \end{bmatrix}$

$$AB = \begin{bmatrix} P & 0_{23} \\ 0_{32} & I \end{bmatrix}\begin{bmatrix} 0_{23} & Q \\ S & 0_{32} \end{bmatrix}$$

$$= \begin{bmatrix} P0_{23} + 0_{23}S & PQ + 0_{23}0_{32} \\ 0_{32}0_{23} + IS & 0_{32}Q + I0_{32} \end{bmatrix}$$

$$= \begin{bmatrix} 0_{23} + 0_{23} & PQ + 0_{22} \\ 0_{33} + S & 0_{32} + 0_{32} \end{bmatrix}$$

$$= \begin{bmatrix} 0_{23} & PQ \\ S & 0_{32} \end{bmatrix}$$

$$PQ = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$$

$$AB = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & -3 & 4 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EXERCISE 3.2 (Page 144)

1.(i) Show that the inverse of a diagonal matrix, with all diagonal elements nonzero, is a diagonal matrix.

Sol. Let
$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

be a diagonal matrix of order $n \times n$. We have

$a_{ij} = 0$, for all $i \neq j$ and $a_{ij} \neq 0$, for all $i = j$.

Let
$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

be the inverse of A. Then $AB = BA = I_n$.

$$AB = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{13}b_{13} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & a_{nn}b_{n3} & \cdots & a_{nn}b_{nn} \end{bmatrix} = I_n$$

$\Rightarrow \quad a_{ii}\, b_{ii} = 1, \qquad i = 1, 2, \cdots, n$ (1)

and $\quad a_{ii}\, b_{ij} = 0, \qquad i \neq j$ and $i, j = 1, 2, \cdots, n$ (2)

From (1), we have $b_{ii} = (a_{ii})^{-1}$

and from (2), we get $b_{ij} = 0$ as $a_{ii} \neq 0$.

$$\text{Hence } B = A^{-1} = \begin{bmatrix} (a_{11})^{-1} & 0 & 0 & \cdots & 0 \\ 0 & (a_{22})^{-1} & 0 & \cdots & 0 \\ 0 & 0 & (a_{23})^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & (a_{nn})^{-1} \end{bmatrix}$$

showing that the inverse of a diagonal matrix is also a diagonal matrix.

1.(ii) Show that the inverse of a scalar matrix is a scalar matrix.

Sol. Let $A = \begin{bmatrix} h & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & h \end{bmatrix}$

be a scalar matrix of order n, $h \neq 0$. Let $B = [a_{ij}]_{n\times n}$ be inverse of A. Then $AB = I_n$ i.e.,

$$\begin{bmatrix} h & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & h \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = I_n$$

$$\text{or} \quad \begin{bmatrix} ha_{11} & ha_{12} & \cdots & ha_{1n} \\ ha_{21} & ha_{22} & \cdots & ha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ha_{n1} & ha_{n2} & \cdots & ha_{nn} \end{bmatrix} = I_n$$

$\Rightarrow \quad ha_{ii} = 1 \qquad$ for $i = 1, 2, \cdots, n$

and $\quad ha_{ij} = 0 \qquad$ for $i \neq j$ and $i, j = 1, 2, \cdots, n$

Thus $a_{ii} = \dfrac{1}{h}$ since $h \neq 0$ and $a_{ij} = 0$ if $i \neq j$

$$\text{Hence } B = \begin{bmatrix} \frac{1}{h} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{h} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{h} \end{bmatrix}$$

which is a scalar matrix.

2. For a nonsingular matrix A, show that

(i) $(A^n)^{-1} = (A^{-1})^n$, here n is a positive integer.

(ii) $(kA)^{-1} = k^{-1}A^{-1}$, k is any nonzero scalar.

(iii) $(A^{-1})^T = (A^T)^{-1}$

(iv) $(\bar{A})^{-1} = (\overline{A^{-1}})$

(v) $\left(\overline{A^T}\right)^{-1} = (\overline{A^{-1}})^T$.

2.(i) $(A^n)^{-1} = (A^{-1})^n$, where n is a positive integer.

Sol. $\quad A^n = A \cdot A \cdot \cdots \cdot A$, n times

$\Rightarrow \quad (A^n)^{-1} = (A \cdot A \cdot \cdots \cdot A)^{-1} = A^{-1} \cdot \cdots \cdot A^{-1} \cdot A^{-1}$, n times

$= (A^{-1})^n$.

Here we have used the extended form of the following:

$(AB)^{-1} = B^{-1} \cdot A^{-1}$.

2.(ii) $(kA)^{-1} = k^{-1}A^{-1}$, k is any nonzero scalar.

Sol. Consider $\quad (kA)(k^{-1}A^{-1}) = kk^{-1}(AA^{-1}) = 1\,I = I.$ (1)

Again, $\quad (kA^{-1})(kA) = k^{-1}k(A^{-1}A) = 1\,I = I.$ (2)

From (1) and (2), we have

$(kA)(kA^{-1}) = (k^{-1}A^{-1})(kA) = I$

Hence $\quad (kA)^{-1} = k^{-1}A^{-1}$

2.(iii) $(A^{-1})^T = (A^T)^{-1}$

Sol. We have $(A^{-1})^T A^T = (AA^{-1})^T$, as $(AB)^T = B^T A^T$
$= I^T = I$ (1)

Again, $A^T (A^{-1})^T = (A^{-1}A)^T$, as $(AB)^T = B^T A^T$
$= I^T = I$ (2)

From (1) and (2), we get

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I$$

Hence $(A^{-1})^T = (A^T)^{-1}$

2.(iv) $(\bar{A})^{-1} = \overline{(A^{-1})}$

Sol. $\bar{A}\,\overline{(A^{-1})} = \overline{(AA^{-1})}$, as $\overline{(AB)} = \bar{A}\,\bar{B}$ $= \bar{I} = I$ (1)

Again, $\overline{(A^{-1})}\,\bar{A} = \overline{(A^{-1}A)}$, as $\overline{(AB)} = \bar{A}\,\bar{B}$ $= \bar{I} = I$ (2)

From (1) and (2), we have

$$\bar{A}\,\overline{(A^{-1})} = \overline{A^{-1}}\,\bar{A} = I$$

Hence $(\bar{A})^{-1} = \overline{(A^{-1})}$.

2.(v) $(\overline{A^T})^{-1} = (\overline{A^{-1}})^T$.

Sol. We have

$$(\overline{A^T})(\overline{A^{-1}})^T = (\bar{A})^T(\overline{A^{-1}})^T, \text{ as } (\overline{A^T}) = (\bar{A})^T$$
$$= (\overline{A^{-1}}\,\bar{A})^T = (\overline{A^{-1}A})^T, \text{ as } (\bar{A}\bar{B}) = \overline{AB}$$
$$= (\bar{I})^T = I \quad (1)$$

Again,

$$(\overline{A^{-1}})^T(\overline{A^T}) = (\overline{A^{-1}})^T(\bar{A})^T, \text{ as } (\overline{A^T}) = (\bar{A})^T$$
$$= (\bar{A}\,\overline{A^{-1}})^T = (\overline{AA^{-1}})^T, \text{ as } (\bar{A}\bar{B}) = \overline{AB}$$
$$= (\bar{I})^T = I \quad (2)$$

From (1) and (2), we obtain

$$(\overline{A^T})(\overline{A^{-1}})^T = (\overline{A^{-1}})^T(\bar{A})^T = I$$

Hence $(\overline{A^T})^{-1} = (\overline{A^{-1}})^T$.

3. If A is invertible and $AB = 0$, then show that $B = 0$.

Sol. Since A is invertible, so $AA^{-1} = I, A^{-1}A = I$.

Now $AB = 0$

$\Rightarrow A^{-1}(AB) = A^{-1}0 = 0$

or $(A^{-1}A)B = 0$, by the Associative Law

or $IB = 0$

or $B = 0$, since $IB = B$.

4. Let A and B be distinct $n \times n$ matrices with real entries. If $AB^2 = BA^2$ and $A^3 = B^3$, show that $A^2 + B^2$ is not invertible.

Sol. Suppose $A^2 + B^2$ is invertible and C is its inverse. Then

$$I = (A^2 + B^2)C.$$

We have

$$A = AI = A(A^2 + B^2)C = A^3C + AB^2C$$
$$= B^3C + AB^2C, \text{ since } A^3 = B^3$$
$$= (B + A)B^2C = (A + B)B^2C \quad (1)$$

and

$$B = BI = B(A^2 + B^2)C = BA^2C + B^3C$$
$$= AB^2C + B^3C, \text{ since } BA^2 = AB^2$$
$$= (A + B)B^2C \quad (2)$$

From (1) and (2), we have $A = B$ contrary to the hypothesis. Thus $A^2 + B^2$ is not invertible.

5. Find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We apply an elementary row operation on A and simultaneously the same row operation on I_3.

$A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$A \overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 - kR_3$	$I_3 \overset{R}{\sim} \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 - kR_3$

Hence $A^{-1} = \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5.(ii) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \text{by } R_{13} \qquad I_3 \underset{\sim}{R} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{by } R_{23} \qquad \underset{\sim}{R} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{by } R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{by } -R_2 \qquad \underset{\sim}{R} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{by } -R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{by } -R_3 \qquad \underset{\sim}{R} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \text{by } -R_3$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

5.(iii) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \text{by } -R_1 \qquad I_3 \underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{by } -R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 4 & -2 & 5 \end{bmatrix} \text{by } R_2-2R_1 \qquad \underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{by } R_2-2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 6 & -7 \end{bmatrix} \text{by } R_3-4R_1 \qquad \underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \text{by } R_3-4R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 1 & -1 \end{bmatrix} \text{by } R_3-R_2 \qquad \underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \text{by } R_3-R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 5 & -6 \end{bmatrix} \text{by } R_{23} \qquad \underset{\sim}{R} \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{by } R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 5 & -6 \end{bmatrix} \text{by } R_1+2R_2 \qquad \underset{\sim}{R} \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{by } R_1+2R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \text{by } R_3-5R_2 \qquad \underset{\sim}{R} \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 1 \\ -8 & 6 & -5 \end{bmatrix} \text{by } R_3-5R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \text{by } -R_2 \qquad \underset{\sim}{R} \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 1 \\ 8 & -6 & 5 \end{bmatrix} \text{by } -R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{by } R_2+R_3 \qquad \underset{\sim}{R} \begin{bmatrix} 3 & -2 & 2 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} \text{by } R_2+R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{by } R_1-R_3 \qquad \underset{\sim}{R} \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} \text{by } R_1-R_3$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

5.(iv) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$A \underset{\sim}{R} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ by } R_3 - 2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ by } R_1 - 2R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ by } R_1 - R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ by } R_2 + 2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ by } -R_1 \text{ and } R_3 + R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ by } R_{23}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_3 - 2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_1 - 2R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 5 & -1 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_1 - R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 5 & -1 & -2 \\ 10 & -1 & -4 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_2 + 2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} -5 & 1 & 2 \\ 10 & -1 & -4 \\ 8 & -1 & -3 \end{bmatrix} \text{ by } -R_1 \text{ and } R_3 + R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 8 & -1 & -3 \\ 10 & -1 & -4 \\ -5 & 1 & 2 \end{bmatrix} \text{ by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \text{ by } R_{23}$$

Thus $A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

5.(v) $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}, (i=\sqrt{1})$

Sol.

$$A = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \text{ by } R_{12}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & i \\ 0 & 2 \end{bmatrix} \text{ by } R_2 - iR_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \text{ by } \frac{1}{2}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ by } R_1 - iR_2$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ by } R_{12}$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 1 \\ 1 & -i \end{bmatrix} \text{ by } R_2 - iR_1$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{-i}{2} \end{bmatrix} \text{ by } \frac{1}{2}R_2$$

$$\begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{bmatrix} \text{ by } R_1 - iR_2$$

Thus $A^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{bmatrix}$

5.(vi) $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}, (i=\sqrt{-1})$

Sol.

$$A = \begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 2 \\ i & -1 & 2i \end{bmatrix} \text{ by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ i & -1 & 2i \end{bmatrix} \text{ by } -R_1$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ by } -R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 4 \\ i & -1 & 2i \end{bmatrix} \text{ by } R_2 - 2R_1 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \text{ by } R_2 - 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 4 \\ 0 & -1 & 3i \end{bmatrix} \text{ by } R_3 - iR_1 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & i \end{bmatrix} \text{ by } R_3 - iR_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 4 \\ 0 & 1 & -3i \end{bmatrix} \text{ by } -R_3 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -i \end{bmatrix} \text{ by } -R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -3i \end{bmatrix} \text{ by } \frac{1}{4}R_2 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ -1 & 0 & -i \end{bmatrix} \text{ by } \frac{1}{4}R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -3i \end{bmatrix} \text{ by } R_1 + R_2 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \\ -1 & 0 & -i \end{bmatrix} \text{ by } R_1 + R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ by } R_3 + 3iR_2 \qquad \underset{\sim}{R}\begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \end{bmatrix} \text{ by } R_3 + 3iR_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ by } R_{23} \qquad \underset{\sim}{R}\begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \text{ by } R_{23}$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

5.(vii) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } R_{13} \qquad \underset{\sim}{R}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } R_{13}$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.(viii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix}$

Sol. We partition the given matrix as under:

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{array}\right]$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

Thus $A = \begin{bmatrix} I_3 & 0_{3\times 2} \\ 0_{2\times 3} & P \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ $\quad\Big|\quad$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ by $R_2 - 3R_1$ $\quad\Big|\quad$ $\underset{\sim}{R} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ by $R_2 - 3R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by $R_1 - 2R_2$ $\quad\Big|\quad$ $\underset{\sim}{R} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ by $R_1 - 2R_2$

Hence $P^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$. Also $I^{-1} = I$.

Now, $\begin{bmatrix} I_3 & 0_{3\times 2} \\ 0_{2\times 3} & P \end{bmatrix}\begin{bmatrix} I_3 & 0_{3\times 2} \\ 0_{2\times 3} & P^{-1} \end{bmatrix} = \begin{bmatrix} I_3 + 0_{3\times 2}\,0_{2\times 3} & I_3 0_{3\times 2} + 0_{3\times 2}P^{-1} \\ 0_{2\times 3}I_3 + P0_{2\times 3} & 0_{2\times 3}\,0_{3\times 2} + PP^{-1} \end{bmatrix}$

$$= \begin{bmatrix} I_3 + 0_{33} & 0_{3\times 2} + 0_{3\times 2} \\ 0_{2\times 3}I_3 + 0_{2\times 3} & 0_{2\times 2} + I_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_3 & 0_{3\times 2} \\ 0_{2\times 3} & I_2 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} I_3^{-1} & 0_{3\times 2} \\ 0_{2\times 3} & P^{-1} \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & -3 & 1 \end{array}\right]$

Alternatively

$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix}$ $\quad\Big|\quad$ $I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$A \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ by $R_5 - 3R_4$ $\quad\Big|\quad$ $I_5 \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$ by $R_5 - 3R_4$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ by $R_4 - 2R_5$ $\quad\Big|\quad$ $\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$ by $R_4 - 2R_5$

Hence $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$

6. Reduce each of the following matrices into the indicated form:

(i) $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ reduced echelon form

Sol. $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$ by $R_2 - 2R_1$ and $R_3 - 3R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 1 & 1 & -2 \end{bmatrix}$ by $R_3 - 2R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 0 & -7 & 10 \\ 0 & 1 & 1 & -2 \end{bmatrix}$ by $R_1 + 2R_3$ and $R_2 - 3R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -7 & 10 \end{bmatrix}$ by R_{23}

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \text{ by } -\frac{1}{7}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \text{ by } R_1 - 5R_3 \text{ and } R_2 - R_3$$

which is the required reduced echelon form of the given matrix.

6.(ii) $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}$ reduced echelon form

Sol. $$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 0 & 2 & 6 & -4 \end{bmatrix} \text{ by } R_3 - R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 0 & -7 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 - R_1 \text{ and } R_3 - 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 2 & 0 & -7 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{12}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } \frac{1}{2}R_1$$

is the required reduced echelon form of the given matrix.

6.(iii) $\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 3 \end{bmatrix}$ echelon form

Sol. $$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 0 \end{bmatrix} \text{ by } R_2 - 2R_1 \text{ and } R_3 - 3R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{12}{5} & 0 \end{bmatrix} \text{ by } \frac{1}{3}R_2 \text{ and } \frac{1}{5}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{2}{5} & -\frac{1}{3} \end{bmatrix} \text{ by } R_3 - R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{5}{6} \end{bmatrix} \text{ by } -\frac{5}{2}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}, \text{ by } R_2 + 2R_3 \text{ and } R_1 - 2R_3$$

is the required echelon form of the given matrix.

6.(iv) $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$ reduced echelon form.

Sol. $$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 4 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}, \text{ by } R_3 - 2R_1 \text{ and } R_4 - 4R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 + R_2 \text{ and } R_4 + R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } \frac{1}{11}R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_1 - 3R_2$$

is the required reduced echelon form of the given matrix.

7. Show that

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \underset{\sim}{R} I_3.$$

Sol.
$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \text{ by } R_2 - 3R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -1 \end{bmatrix}, \text{ by } R_{23}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix}, \text{ by } R_1 - 2R_2 \text{ and } R_3 + 5R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } \frac{1}{9}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } R_1 + 3R_3 \text{ and } R_2 - 2R_3$$

$= I_3.$

8. Find the rank of each of the following matrices:

(i)
$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$$

Sol.
$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -16 \\ 0 & 9 \end{bmatrix}, \text{ by } R_3 - 5R_1 \text{ and } R_4 + 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \text{ by } \frac{-1}{2}R_2, \frac{-1}{16}R_3 \text{ and } \frac{1}{9}R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ by } R_3 - R_2 \text{ and } R_4 - R_2$$

Thus the rank of the given matrix is 2.

8.(ii)
$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$$

Sol.
$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{bmatrix}, \text{ by } R_2 - 2R_1, R_3 + 2R_1 \text{ and } R_4 + R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 6 & -5 \end{bmatrix}, \text{ by } -\frac{1}{3}R_2 \text{ and } \frac{1}{3}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -17 \end{bmatrix}, \text{ by } R_3 - R_2 \text{ and } R_4 - 6R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{by } -\frac{1}{3}R_3 \text{ ands } -\frac{1}{17}R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{by } R_4 - R_3$$

The given matrix is row equivalent to a matrix in echelon form, having three nonzero rows. Hence the rank of the given matrix is 3.

8.(iii) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Sol. $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}, \quad \text{by } R_2 - R_1, R_3 - 2R_1 \text{ and } R_4 - 3R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_3 + 3R_2 \text{ and } R_4 + R_2$$

The given matrix is row equivalent to a matrix in echelon form, having two nonzero rows. Hence the rank of the given matrix is 2.

8.(iv) $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

Sol. $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 4 & -1 & -1 \\ 0 & 1 & 1 & -4 & 5 \end{bmatrix}, \quad \text{by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{bmatrix}, \quad \text{by } R_3 - R_2 \text{ and } R_4 - R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{by } R_4 + 2R_3$$

The given matrix is row equivalent to a matrix in echelon form, having three nonzero rows. Hence the rank of the given matrix is 3.

8.(v) $\begin{bmatrix} ar^{(1-1)n} & ar^{(1-1)n+1} & \cdots & ar^{(1-1)n+(n-1)} \\ ar^{(2-1)n} & ar^{(2-1)n+1} & \cdots & ar^{(2-1)n+(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ ar^{(n-1)n} & ar^{(n-1)n+1} & \cdots & ar^{(n-1)n+(n-1)} \end{bmatrix}$, ($a$ and r are nonzero).

Sol. The given matrix is row equivalent to

$$\begin{bmatrix} a & ar & \cdots & ar^{n-1} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{by } R_2 - r^n R_1, R_3 - r^{2n} R_1, \cdots \text{ and } R_n - r^{(n-1)n} R_1.$$

This is an echalon matrix with one nonzero row.

Hence Rank $A = 1$.

9. Reduce each of the following matrices to the canonical form. In each case also find nonsingular matrices P and Q.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Sol.

	A	I_2 Row Operations	I_2 Column Operations
	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
by R_{12};	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
by $R_2 - 2R_1$;	$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
by $C_2 - 2C_1$:	$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$
by $-\frac{1}{3}C_2$:	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$$

$$PAQ = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9.(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol.

	A	I_2 Row Operations	I_3 Column Operations
	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $R_1 - R_2$;	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $R_2 - R_1$;	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $C_3 - C_2$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAQ = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [\, I_2 \;\; O_{2\times 1} \,]$$

9.(iii) $\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ We write $A = I_3\ A\ I_3$

and perform the same elementary row or column operation on A and indicated identity matrix simultaneously.

	A	I_3 Row Operations	I_3 Column Operations
	$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $R_2 - 2R_1$:	$\begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $C_2 + C_1$:	$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $C_3 - 3C_1$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $R_2 + R_3$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $R_3 - 3R_2$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 11 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
by $C_3 + 3C_2$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$
by $\frac{1}{11}C_3$:	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3/11 \\ 0 & 0 & 1/11 \end{bmatrix}$

Hence the required normal form of A is

$$I_3,\ P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix},\quad Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3/11 \\ 0 & 0 & 1/11 \end{bmatrix}$$

9.(iv) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ We write $A = I_3 A I_4$

and perform the same elementary row or column operation on A and the identity matrix simultaneously.

	A	I_3 Row Operations	I_4 Column Operations
	$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
by $C_2 - 2C_1$:	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & -2 & 1 & 2 \\ -2 & 7 & 2 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
by $C_4 + C_1$:	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
by $R_2 - 3R_1$ and $R_3 + 2R_1$:	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
by $C_2 + 2C_3$ and $C_4 - 5C_3$:	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & -7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
by R_{23}:	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 2 & -7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

by $R_2 - 2R_3$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & -7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & -2 & 1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

by $\frac{1}{11}C_2$ and $\frac{1}{7}C_4$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & -2 & 1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2/11 & 0 & 1/7 \\ 0 & 1/11 & 0 & 0 \\ 0 & 2/11 & 1 & -5/7 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$

by $C_4 + C_2$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & -2 & 1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2/11 & 0 & -3/77 \\ 0 & 1/11 & 0 & 1/11 \\ 0 & 2/11 & 1 & -41/77 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$

Hence the required normal form is $[\,I_3 \;\; 0_{3\times1}\,]$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 8 & -2 & 1 \\ -3 & 1 & 0 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & -2/11 & 0 & -3/77 \\ 0 & 1/11 & 0 & 1/11 \\ 0 & 2/11 & 1 & -41/77 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

10. (i) Using the row operations, show that the matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \text{ has no inverse.}$$

Sol. $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & -12 & 12 \end{bmatrix}$, by $R_2 - R_1$ and $R_3 - 5R_1$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{by } R_3 - 3R_2$$

Thus A is singular and so it has no inverse.

10.(ii) If $A^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}$, compute $(AB)^{-1}$

Sol. Since $(AB)^{-1} = B^{-1}A^{-1}$, we have

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 5 \\ -2 & 2 & -8 \\ 0 & 5 & -3 \end{bmatrix}$$

11. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be $n \times 1$ matrices.

Then the products $x^T y$ and xy^T are 1×1 and $n \times n$ matrices respectively. The products $x^T y$ and xy^T are respectively called the **inner** (or **scalar**) and **outer products** of x and y.

(i) Write the inner and outer products of x and y.

(ii) Show that the inner product of x and y is equal to the inner product of y and x (i.e., $x^T y = y^T x$).

(iii) Prove that rank $(x y^T) = 1$.

Sol. (i) $x^T y = [\,x_1 \;\; x_2 \;\; \cdots \;\; x_n\,] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\,x_1y_1 + x_2y_2 + \cdots + x_ny_n\,]$

$$xy^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [\,y_1 \;\; y_2 \;\; \cdots \;\; y_n\,] = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \cdots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$$

(ii) $x^T y = [\,y_1 \;\; y_2 \;\; \cdots \;\; y_n\,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= [\,y_1x_1 + y_2x_2 + \cdots + y_nx_n\,] = x^T y$$

(iii) $xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \cdots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

by $R_2 - \frac{x_2}{x_1}R_1, R_3 - \frac{x_3}{x_1}R_1, \cdots, R_n - \frac{x_n}{x_1}R_1$

$\Rightarrow$ Rank $(xy^T) = 1$.

12. Let A and B be idempotent matrices i.e. $A^2 = A$, $B^2 = B$.

Show that:

(i) if $AB = BA$ then AB is idempotent.

(ii) if A^T is idempotent so is A.

Is the sum of two idempotent matrices idempotent? Justify your answer.

Sol.(i) $(AB)^2 = (AB)(AB)$
$= ABAB$, by the Associative Law
$= AABB$, since $AB = BA$ (given)
$= A^2B^2 = AB$ as $A^2 = A$ and $B^2 = B$

Thus AB is idempotent.

12.(ii) $(A^T)^2 = A^T$ since A^T is idempotent.
$A^2 = AA = (A^T)^T(A^T)^T$
$= B^T B^T$, where $B = A^T = (BB)^T$
$= (B^2)^T = ((A^T)^2)^T = (A^T)^T = A$

showing that A is idempotent.

Let A and B be two idempotent matrixes of the same order.
Then $A^2 = A$, $B^2 = B$ and $A + B$ is defined.
Now,

$$(A+B)^2 = (A+B)(A+B)$$
$$= A^2 + AB + BA + B^2$$
$$= A + B + AB + BA$$
$$= A + B \text{ if and only if } AB = -BA.$$

Therefore $A + B$ is not idempotent in general.

13. If A is an $n \times n$ nilpotent matrix, show that $I_n - A$ is nonsingular.

Sol. A is a nilpotent square matrix of order n and suppose $A^p = 0$.

$$(I_n - A)(I_n - A)^{-1} = (I_n - A)(I_n + A + A^2 + \cdots + A^p + \cdots)$$

by the binomial theorem.

$$= (I_n - A)(I_n + A + A^2 + \cdots + A^{p-1})$$

since $A^p = 0$.

$$= (I_n + A + A^2 + \cdots + A^{p-1}) - (A + A^2 + \cdots + A^{p-1} + A^p)$$
$$= I_n$$

Again,

$$(I_n - A)^{-1}(I_n - A)$$
$$= (I_n + A + A^2 + \cdots + A^p + \cdots)(I_n - A)$$
$$= (I_n + A + A_2 + \cdots + A^{p-1})(I_n - A)$$
$$= (I_n + A + A_2 + \cdots + A^{p-1}) - (A + A^2 + \cdots + A^{p-1} + A^p)$$
$$= I_n$$

Hence $(I_n - A)(I_n - A)^{-1} = (I_n - A)^{-1}(I_n - A) = I_n$

Thus $(I_n - A)^{-1}$ is the inverse of $I_n - A$ and so $I_n - A$ is nonsingular.