Chapter 5

DETERMINANTS

EXERCISE 5.1 (Page 201)

1. Let M_2 be the set of all 2×2 matrices. Set up the transformation $A \to \det A$, $A \in M_2$. What is the range of this mapping? Is the mapping one-to-one?

Sol. Let $f : A \to \det A$, for all $A \in M_2$ be defined by

$$f(A) = \det A.$$

and let the field for all members of M_2 be the set of complex numbers **C**. Then the range of f will be **C**. The set of real numbers R is a subset of the set of complex numbers **C**. Thus if R is taken as the field for M_2, then the range of the mapping f will be R.

The mapping f cannot be one-to-one, as for some $A, B \in M_2$,

$$A \neq B \Rightarrow \det A = \det B.$$

We justify this by means of an example:

Let $$A = \begin{bmatrix} 5 & 4 \\ 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 10 \\ 3 & 7 \end{bmatrix}$$

then $$\det A = \begin{vmatrix} 5 & 4 \\ 7 & 8 \end{vmatrix} = 40 - 28 = 12,$$

$$\det B = \begin{vmatrix} 6 & 10 \\ 3 & 7 \end{vmatrix} = 42 - 30 = 12.$$

Clearly, $A \neq B$ but $\det A = \det B$.

2. For 2 × 2 matrices A and B which of the following equations hold?

(i) $\det(A+B) = \det A + \det B$

Sol. (i) We disprove the result by a counter example.

Let $A = \begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 11 \end{bmatrix}$$

$$\det(A+B) = \begin{vmatrix} 3 & -3 \\ 3 & 11 \end{vmatrix} = 33 + 9 = 42.$$

Also $\det A = \begin{vmatrix} 1 & -4 \\ 3 & 5 \end{vmatrix} = 5 + 12 = 17$ and $\det B = \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} =$

Hence $\det A + \det B = 17 + 12 = 29 \neq 42 = \det(A+B)$.

2.(ii) $\det(A+B)^2 = (\det(A+B))^2$

Sol. We know that for any $n \times n$ matrices X, Y

$$\det(X \cdot Y) = \det X \cdot \det Y, \quad \text{by (5.18)}$$

Thus for 2 × 2 matrices A and B, we have

$$\det(A+B)^2 = \det((A+B)\cdot(A+B)) = \det(A+B)\cdot\det(A+B) = (\det(A+B))^2$$

2.(iii) $\det(A+B)^2 = \det(A^2 + B^2)$

Sol. We give a counter example to disprove this result. Let

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 5 & 4 \end{bmatrix}$$

and $\det(A+B)^2 = \begin{vmatrix} 9 & 0 \\ 5 & 4 \end{vmatrix} = 36.$

Now $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

and $B^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$.

Hence $A^2 + B^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

$$\det(A^2 + B^2) = \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} = 2 \neq 36 = \det(A+B)^2$$

Thus $\det(A+B)^2 = \det(A^2 + B^2)$ does not hold.

2.(iv) Let A and B be as in **2(iii)** above.

Then $AB = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$

and so $2AB = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}$

Hence $A^2 + 2AB + B^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 6 & 5 \end{bmatrix}$

and $\det(A^2 + 2AB + B^2) = \begin{vmatrix} 8 & 5 \\ 6 & 5 \end{vmatrix} = 40 - 6 = 34.$

$$\det(A+B)^2 = \begin{vmatrix} 9 & 0 \\ 5 & 4 \end{vmatrix} = 36 \neq \det(A^2 + 2AB + B^2)$$

Consequently, the result $\det(A+B)^2 = \det(A^2 + 2AB + B^2)$ does not hold.

3. Evaluate the following determinants:

(i) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$

Sol. (i) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & -1 \\ 5 & 6 & -3 \end{vmatrix}$, by $C_3 - 2C_1$

$= \begin{vmatrix} 4 & -1 \\ 6 & -3 \end{vmatrix}$, expanding by the first row

$= -12 + 6 = -6.$

3.(ii) $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix}$

Sol. $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -5 & 6 & 4 \\ -7 & 3 & 3 \end{vmatrix}$, by $C_1 - 2C_3$ and $C_2 + C_3$

$= \begin{vmatrix} -5 & 6 \\ -7 & 3 \end{vmatrix}$, expanding by the first row

$= -15 + 42 = 27$

3.(iii) $\begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 10 & -5 & 5 \end{vmatrix}$

Sol. $\begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 10 & -5 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 2 & 6 & -8 \\ 10 & 5 & -5 \end{vmatrix}$, by $C_2 + C_1$ and $C_3 - C_1$

$= 6\begin{vmatrix} 6 & -8 \\ 5 & -5 \end{vmatrix}$, expanding by the first row

$= 6(-30 + 40) = 60.$

4. Evaluate

(i) $\begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$

Sol. (i) $\begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 1 & -1 & 5 & 0 \\ -2 & 4 & 0 & 5 \end{vmatrix}$, by $R_3 - R_1$

$= \begin{vmatrix} 0 & 5 & -12 & 4 \\ 0 & 11 & -38 & 10 \\ 1 & -1 & 5 & 0 \\ 0 & 2 & 10 & 5 \end{vmatrix}$, by $R_1 - 2R_3$, $R_2 - 7R_3$ and $R_4 + 2R_3$

$= \begin{vmatrix} 5 & -12 & 4 \\ 11 & -38 & 10 \\ 2 & 10 & 5 \end{vmatrix}$, expanding by the first column

$= 5(-190 - 100) + 12(55 - 20) + 4(110 + 76)$

$= 5(-290) + 12(35) + 4(186)$

$= -1450 + 420 + 744 = -1450 + 1164 = -286.$

4.(ii) $\begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix}$

Sol. $\begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix} = -2\begin{vmatrix} 7 & 5 & 2 \\ 4 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix}$, expanding by R_3

$= -2\,[7(4 - 3) - 5(16 - 1) + 2(12 - 1)]$

$= -2\,[7 - 75 + 22] = -2(46) = -92.$

4.(iii) $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & -3 & 1 \\ 0 & -7 & 3 & 1 \end{vmatrix}$, by $R_3 - R_1$

$= \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 1 \\ -7 & 3 & 1 \end{vmatrix}$, expanding by R_1

$= \begin{vmatrix} 1 & 0 & 0 \\ 5 & 2 & -4 \\ -7 & -4 & 8 \end{vmatrix}$, by $C_2 + C_1$ and $C_3 - C_1$

$= 16 - 16 = 0$

4.(iv) $\begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

Sol. $\begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix} = -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ 1 & 92 & 84 & 1 \\ 2 & 185 & 100 & 2 \\ 4 & 270 & 196 & 4 \end{vmatrix}$, by taking -6 common from C_4

$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ -8 & -1 & 72 & 0 \\ -16 & -1 & 76 & 0 \\ -32 & -102 & 148 & 0 \end{vmatrix}$, by $R_2 - R_1$; $R_3 - 2R_1$ and $R_4 - 4R_1$

$= 6 \begin{vmatrix} -8 & -1 & 72 \\ -16 & -1 & 76 \\ -32 & -102 & 148 \end{vmatrix}$, expanding by C_4

$= 192 \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$, taking -8 common from C_1; -1 common from C_2 and 4 common from C_3

$= 192 \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -17 \\ 4 & 98 & -35 \end{vmatrix}$ b $C_2 - C_1$ and $C_3 - 18C_1$

$= 192(35 + 17 \times 98) = 192(35 + 1666) = 192(1701) = 326592$

4.(v) $\begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \end{vmatrix}$ by $C_4 + C$

$= 0$

Since every element of C_4 is zero, so the value of the determinant in the last step is zero.

4.(vi) $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$ in factorised form.

Sol. The given determinant Δ is a circulant. By the elementary row operation, $R_1 + wR_4 + w^2R_3 + w^3R_2$. ($w^4 = 1$), we have

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}, \text{ where}$$

$$\alpha = a + bw + cw^2 + dw^3$$

$$\beta = b + cw + dw^2 + aw^3 = w^3(a + bw + cw^2 + dw^3)$$

$$\gamma = c + wd + aw^2 + bw^3 = w^2(a + bw + cw^2 + dw^3)$$

$$\delta = aw + bw^2 + cw^3 + d = w(a + bw + cw^2 + dw^3)$$

Thus $a + bw + cw^2 + dw^3$ is a common factor in the first row.

$$\Delta = (a + bw + cw^2 + dw^3) \cdot \begin{vmatrix} 1 & w^3 & w^2 & w \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$$

Since $w^4 = 1$, and there are four 4th roots of 1, viz. $1, -1, i, -i$, replacing w by these four roots, we see that $a + b + c + d$, $a - b + c - d$, $a + ib - c - d$, $a - ib - c + id$ are factors of Δ.

Hence $\Delta = k(a + b + c + d)(a - b + c - d)(a + ib - c - id)(a - ib - c + id)$

Equating the coefficient of a^4 on both sides, we find $k = 1$. Hence

$$\Delta = \Pi\,(a + bw + cw^2 + dw^3)$$

where $w = 1, -1, i, -i$.

5. Without expanding show that

(i) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$$

Sol. (i) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix}, \text{ as } \det A = \det A^T$$

$$= -\begin{vmatrix} b & e & h \\ a & d & g \\ c & f & k \end{vmatrix}, \text{ by } R_{12}$$

$$= \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}, \text{ by } C_{12}$$

5.(ii) $$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

Sol. Let $$A = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}, \text{ as } A = A^T$$

$$= (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}, \text{ by taking } -1 \text{ common from each row}$$

$$= -A$$

Thus $2A = 0$ implies $A = 0$.

5.(iii) $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0.$$

Sol. $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & c & 1 \\ a+b+c & a & 1 \\ a+b+c & b & 1 \end{vmatrix}, \text{ by } C_1 + C_2$$

$$= (a+b+c)\begin{vmatrix} 1 & c & 1 \\ 1 & a & 1 \\ 1 & b & 1 \end{vmatrix}, \text{ Taking } a+b+c \text{ common from } C_1$$

$$= 0, \text{ as } C_1 = C_3.$$

6. Prove that each of following determinants vanishes:

(i) $$\begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Sol. (i) $$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= (abc)\begin{vmatrix} \frac{bc}{abc} & \frac{ca}{abc} & \frac{ab}{abc} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}, \text{ multiplying the first row by } \frac{1}{abc} \text{ and dividing the det by } \frac{1}{abc}$$

$$= (abc)\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = abc(0) = 0, \text{ as } R_1 = R_2$$

6.(ii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Sol. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$, by $R_1 + (R_2 + R_3)$

$= 0.$

6.(iii) $\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix}$

Sol. $\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = (1/abc) \begin{vmatrix} a^2 & a^3 & a^2 \\ b^2 & b^3 & b^2 \\ c^2 & c^3 & c^2 \end{vmatrix}$, by multiplying C_3 by abc and dividing the det by abc

$= 0$, as $C_1 = C_3$

6.(iv) $\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix}$

Sol. $\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = \begin{vmatrix} \sin^2\theta + \cos^2\theta & 1 & \cos\theta \\ \sin^2\theta + \cos^2\phi & 1 & \cos^2\phi \\ \sin^2\psi + \cos^2\psi & 1 & \cos^2\psi \end{vmatrix}$, by $C_1 + C_3$

$= \begin{vmatrix} 1 & 1 & \cos^2\theta \\ 1 & 1 & \cos^2\phi \\ 1 & 1 & \cos^2\psi \end{vmatrix} = 0$, as $C_1 = C_2$

6.(v) $\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$

Sol. $\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$

$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha - \sin^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta - \sin^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma - \sin^2\gamma \end{vmatrix}$, by $C_3 - C_1$

$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos 2\alpha \\ \sin^2\beta & \cos 2\beta & \cos 2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos 2\gamma \end{vmatrix} = 0$, (as $C_2 = C_3$)

6.(vi) $\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix}$

Sol. $\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix}$

$= \begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) - \sin\alpha\cos\delta - \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \sin(\beta+\delta) - \sin\beta\cos\delta - \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) - \sin\gamma\cos\delta - \cos\gamma\sin\delta \end{vmatrix}$

by $C_3 - (C_1 \sin\delta + C_2 \cos\delta)$

$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} = 0$

6.(vii) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & 1-\cos^2\alpha & \cos(\alpha+\beta)-\cos\alpha\cos\beta \\ \cos\beta & \cos(\alpha+\beta)-\cos\alpha\cos\beta & 1-\cos^2\beta \end{vmatrix}$

by $C_2 - C_1\cos\alpha$ and $C_3 - C_1\cos\beta$

$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & \sin^2\beta \end{vmatrix}$

$= -\sin\alpha\sin\beta \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & -\sin\beta & -\sin\beta \end{vmatrix}$

$= 0$ as C_2 and C_3 are identical.

6.(viii) $\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix}$

Sol. $\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = \begin{vmatrix} 2ab & a^2+b^2 & ab \\ 2cd & c^2+d^2 & cd \\ 2gh & g^2+h^2 & gh \end{vmatrix}$, by $C_1 - C_2$

$= 2\begin{vmatrix} ab & a^2+b^2 & ab \\ cd & c^2+d^2 & cd \\ gh & g^2+h^2 & gh \end{vmatrix} = 0$, as $C_1 = C_3$

6.(ix) $\begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix}$

Sol. $\begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix}$

$= \begin{vmatrix} 4 & (a^m-a^{-m})^2 & abc \\ 4 & (b^n-b^{-n})^2 & abc \\ 4 & (c^p-c^{-p})^2 & abc \end{vmatrix}$ by $C_1 - C_2$

$= 4abc\begin{vmatrix} 1 & (a^m-a^{-m})^2 & 1 \\ 1 & (b^n-b^{-n})^2 & 1 \\ 1 & (c^p-c^{-p})^2 & 1 \end{vmatrix} = 0$, as $C_1 = C_3$.

6.(x) $\begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}$

Sol. $\begin{vmatrix} 1/2 & 1 & 0 \\ 1/3! & 1/2! & 1 \\ 1/4! & 1/3! & 1/2! \end{vmatrix} = \begin{vmatrix} 1/2 & 1 & 0 \\ 1/6 & 1/2 & 1 \\ 1/24 & 1/6 & 1/2 \end{vmatrix}$, replacing factorials by their values

$= \frac{1}{2\cdot6\cdot24}\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$, by multiplying R_1 by 2; R_2 by 6; R_3 by 24 and adjusting the det by the factor $\frac{1}{2\cdot6\cdot24}$

$= \frac{1}{2\cdot6\cdot24}\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 6 \\ 0 & 2 & 12 \end{vmatrix}$, by $R_2 - R_1$ and $R_3 - R_1$

$= \frac{6}{2\cdot6\cdot24}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}$, by $C_2 - 2C_1$ and taking out 6 from C_3

$= 0$ as $C_2 = C_3$.

6.(xi) $\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$

where a, b, c are the magnitudes of the sides of a triangle and A is the measure of the angle opposite to the side with magnitude a.

Sol. $\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$

$= \frac{1}{\sin A}\begin{vmatrix} a^2\sin A & b\sin A & c\sin A \\ b\sin^2 A & 1 & \cos A \\ c\sin^2 A & \cos A & 1 \end{vmatrix}$, multiplying C_1 by $\sin A$ and adjusting by the factor $\frac{1}{\sin A}$

$= \frac{1}{\sin A}\begin{vmatrix} (a^2-b^2-c^2)\sin A & b\sin A & c\sin A \\ b\sin^2 A - b - c\cos A & 1 & \cos A \\ c\sin^2 A - b\cos A - c & \cos A & 1 \end{vmatrix}$, by $C_1 - (bC_2 + cC_3)$

$= \frac{1}{\sin A}\begin{vmatrix} -2bc\cos A\sin A & b\sin A & c\sin A \\ -b\cos^2 A - c\cos A & 1 & \cos A \\ -c\cos^2 A - b\cos A & \cos A & 1 \end{vmatrix}$ as $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$= \cot A\begin{vmatrix} -2bc\sin A & b\sin A & c\sin A \\ -b\cos A - c & 1 & \cos A \\ -c\cos A - b & \cos A & 1 \end{vmatrix}$

$= \cot A\begin{vmatrix} 0 & b\sin A & c\sin A \\ 0 & 1 & \cos A \\ 0 & \cos A & 1 \end{vmatrix}$, by $C_1 + c\,C_2 + b\,C_3$

$= 0$

6.(xii) $\begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix}$

Sol. $\begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix} = \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ a+b+c+d & c & d & a & 1 \\ a+b+c+d & d & a & b & 1 \\ a+b+c+d & a & b & c & 1 \\ a+b+c+d & a & d & c & 1 \end{vmatrix}$, by $C_1 + (C_2 + C_3 + C_4)$

$= (a+b+c+d)\begin{vmatrix} 1 & b & c & d & 1 \\ 1 & c & d & a & 1 \\ 1 & d & a & b & 1 \\ 1 & a & b & c & 1 \\ 1 & a & d & c & 1 \end{vmatrix}$

$= 0$, as $C_1 = C_5$

6.(xiii) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

Sol. $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix}$,

by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$

$= \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix}$, by $C_3 - 2C_2$ and $C_4 - 3C_2$

$$= 3\begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0 \text{ as } C_3 = C_4$$

7. Without expansion, prove that

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Sol. $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a & a^2 \\ abc & b & b^2 \\ abc & c & c^2 \end{vmatrix}$, multiplying R_1; R_2 and R_3 by a; b and c respectively and adjusting by the factor $\frac{1}{abc}$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

8. Show that

$$\begin{vmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{vmatrix} = (a-b+c)\,(a+b\omega_1+c\omega_1^2)\,(a+b\omega_2+c\omega_2^2)$$

where ω_1 and ω_2 are cube roots of -1.

Sol. $\Delta = \begin{vmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & -a & -b \\ b & c & -a \end{vmatrix}$, by $(-1)R_2$ and $(-1)R_3$

$$= \begin{vmatrix} a+bw_1+cw_1^2 & b+cw_1-aw_1^2 & c-aw_1-bw_1^2 \\ c & -a & -b \\ b & c & -a \end{vmatrix}$$

by $R_1 + w_1R_3 + w_1^2R_2$ (where w_1 ...)

$$= \begin{vmatrix} a+bw_1+cw_1^2 & -w_1^2(a+bw_1+cw_1^2) & -w_1(a+bw_1+cw_1^2) \\ c & -a & -b \\ b & c & -a \end{vmatrix}$$

$$= (a+bw_1+cw_1^2)\begin{vmatrix} 1 & -w_1^2 & -w_1 \\ c & -a & -b \\ b & c & -a \end{vmatrix}$$

Thus $a + bw_1 + cw_1^2$ is a factor of Δ.

Since -1 and w_2 are other two cube roots of -1, $a + (-1)b + (-1)^2 c$ and $a + bw_2 + cw_2^2$ are also factors of Δ. Hence

$$\Delta = k(a-b+c)(a+bw_1+cw_1^2)(a+bw_2+cw_2^2)$$

where k is a constant since Δ involves terms of third degree. Comparing coefficient of a^3 on both sides, we find $k = 1$. Therefore, $\Delta = (a-b+c)(a+bw_1+cw_1^2)(a+bw_2+cw_2^2)$, where w_1, w_2 are complex cube roots of -1.

9. Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

Sol. $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$

$= \frac{1}{abcd}\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ -ab & ab & -cd & cd \\ -ac & -bd & ac & -bd \\ -ad & bc & bc & ad \end{vmatrix}$, by multiplying C_1; C_2; C_3, C_4 respectively by a; b; c; d and adjusting by the factor $\frac{1}{abcd}$

$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -cd & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$, by $C_1 + (C_2 + C_3 + C_4)$

$= \frac{a^2+b^2+c^2+d^2}{abcd}\begin{vmatrix} ab & -cd & cd \\ bd & ac & -bd \\ -bc & bc & ad \end{vmatrix}$, expanding by C_1

$$= \frac{bcd(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$, by taking b; c; d common from C_1; C_2; C_3 respectively

$$= \frac{a^2+b^2+c^2+d^2}{a}[a(a^2+b^2)+d(ad-bc)+c(bd+ac)]$$

$$= \frac{a^2+b^2+c^2+d^2}{a}[a^3+ab^2+ad^3-dbc+cbd+ac^2)]$$

$$= \frac{a^2+b^2+c^2+d^2}{a}[a(a^2+b^2+d^2+c^2)] = (a^2+b^2+c^2+d^2)^2$$

10. Prove that

(i) $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Sol. (i) $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$, by $C_1 - C_3$, $C_2 - C_3$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & (c+a-b) & b^2 \\ (c-a-b) & (c-a-b) & (a+b)^2 \end{vmatrix}$$ by taking $(a+b+c)$ common from each of C_1 and C_2

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$, by $R_3 - (R_1 + R_2)$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b(c+a) \\ -2b & -2a & 0 \end{vmatrix}$$, by $C_3 + bC_2$

$$= (a+b+c)^2[(b+c-a)\{2ab(c+a)\} + a^2 . 2b(c+a-b)]$$
$$= 2ab(a+b+c)^2[bc+ab+c^2+ac-ac-a^2+ac+a^2-ab]$$
$$= 2ab(a+b+c)^2[c(a+b+c)] = 2abc(a+b+c)^3$$

10.(ii) $$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Sol. $$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{a^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$, by multiplying R_1; R_2; R_3 by c; a; b respectively and adjusting with factor $\frac{1}{abc}$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ b^2-c^2-a^2 & b^2-c^2-a^2 & c^2+a^2 \end{vmatrix}$$, by $C_1 - C_3$ and $C_2 - C_3$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ -2a^2 & -2c^2 & 0 \end{vmatrix}$$, by $R_3 - (R_1 + R_2)$

$$= \frac{1}{abc}[(a^2+b^2-c^2)\{0+2a^2c^2\} + c^2\{2a^2(b^2+c^2-a^2)\}]$$

$$= \frac{1}{abc}[2a^2c^2(a^2+b^2-c^2) + 2a^2c^2(b^2+c^2-a^2)]$$

$$= \frac{2a^2c^2}{abc}[a^2+b^2-c^2+b^2+c^2-a^2]$$

$$= \frac{2a^2c^2}{abc}[2b^2] = \frac{4a^2b^2c^2}{abc} = 4abc$$

11. Prove that

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x-a)^3(x+3a).$$

Sol. $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = \begin{vmatrix} x+3a & x+3a & x+3a & x+3a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$, by $R_1 + (R_2+R_3+R_4)$

$= (x+3a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$, by taking $(x+3a)$ common from the first row

$= (x+3a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & x-a & 0 & 0 \\ a & 0 & x-a & 0 \\ a & 0 & 0 & x-a \end{vmatrix}$, by subtracting C_1 from each of C_2; C_3; C_4

$= (x+3a)(x-a)^3$, as the determinant is lower triangular, so its value is equal to the product of its diagonal elements.

12. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

Sol. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \beta\gamma-\alpha\beta & \gamma\alpha-\alpha\beta & \alpha\beta \end{vmatrix}$ by $C_1 - C_3$ and $C_2 - C_3$

$= (\gamma-\alpha)(\beta-\gamma)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \gamma \\ \beta & -\alpha & \alpha\beta \end{vmatrix}$, by taking $(\gamma-\alpha)$ common from C_1 and $(\beta-\gamma)$ common from C_2

$= (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta) = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$

13. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma).$$

Sol. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \alpha^3-\gamma^3 & \beta^3-\gamma^3 & \gamma^3 \end{vmatrix}$, by $C_1 - C_3$ and $C_2 - C_3$

$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ (\alpha-\gamma)(\alpha^2+\alpha\gamma+\gamma^2) & (\beta-\gamma)(\beta^2+\beta\gamma+\gamma^2) & \gamma^3 \end{vmatrix}$

$= (\alpha-\gamma)(\beta-\gamma)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ \alpha^2+\alpha\gamma+\gamma^2 & \beta^2+\beta\gamma+\gamma^2 & \gamma^3 \end{vmatrix}$,

by taking $(\alpha-\gamma)$ common from C_1; and $(\beta-\gamma)$ common from C_2

$= -(\beta-\gamma)(\gamma-\alpha)[\beta^2+\beta\gamma+\gamma^2-\alpha^2-\alpha\gamma-\gamma^2]$

$= -(\beta-\gamma)(\gamma-\alpha)[-\alpha^2+\beta^2-\alpha\gamma+\beta\gamma]$

$= (\beta-\gamma)(\gamma-\alpha)[\alpha^2-\beta^2+\alpha\gamma-\beta\gamma]$

$= (\beta-\gamma)(\gamma-\alpha)[(\alpha-\beta)(\alpha+\beta)+\gamma(\alpha-\beta)]$

$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$

14. Show that

(i) $$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a-1)^3(a+3)$$

Sol. $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = \begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$, by $R_1 + (R_2+R_3+R_4)$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}, \text{ by taking } (a+3) \text{ common from the first row}$$

$$= (a+3)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a-1 & 0 & 0 \\ 1 & 0 & a-1 & 0 \\ 1 & 0 & 0 & a-1 \end{vmatrix}, \text{ by subtracting } C_1 \text{ from each of } C_2, C_3, C_4$$

$= (a+3)(a-1)^3$,

as the determinant is lower triangular, so its value is equal to the prod[uct] of its diagonal elements.

14.(ii) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Sol. $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

$$= (abcd)\begin{vmatrix} 1/a+1 & 1/a & 1/a & 1/a \\ 1/b & 1/b+1 & 1/b & 1/b \\ 1/c & 1/c & 1+1/c & 1/c \\ 1/d & 1/d & 1/d & 1/d+1 \end{vmatrix}, \text{ taking } a, b, c, d \text{ common from } R_1, R_2, R_3, R_4 \text{ respectively and adjusting its effect by the factor } abcd$$

$$= (abcd)\begin{vmatrix} 1+x & 1+x & 1+x & 1+x \\ 1/b & 1/b+1 & 1/b & 1/b \\ 1/c & 1/c & 1/c+1 & 1/c \\ 1/d & 1/d & 1/d & 1/d+1 \end{vmatrix}, \text{ by } R_1+(R_2+R_3+R_4) \text{ and writing } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=x$$

$$= (abcd)(1+x)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1/b & 1/b+1 & 1/b & 1/b \\ 1/c & 1/c & 1/c+1 & 1/c \\ 1/d & 1/d & 1/d & 1/d+1 \end{vmatrix}, \text{ by taking } (1+x) \text{ common from } R_1$$

$$= (abcd)(1+x)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1/b & 1 & 0 & 0 \\ 1/c & 0 & 1 & 0 \\ 1/d & 0 & 0 & 1 \end{vmatrix}, \text{ by } C_2-C_1, C_3-C_1 \text{ and } C_4-C_1$$

$= (abcd)(1+x)$,

since the determinant is lower triangular so its value is equal to the product of its diagonal elements.

$$= (abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

14.(iii) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2$$

Sol. $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -x & -x & 0 & 1 \\ -x & 0 & y & 1 \\ 1-(1+x)(1-y) & y & y & 1-y \end{vmatrix}, \text{ by } C_1-(1+x)C_4, C_2-C_4 \text{ and } C_3-C_4$$

$$= -\begin{vmatrix} -x & -x & 0 \\ -x & 0 & y \\ -x+y+xy & y & y \end{vmatrix}, \text{ expanding by the first row}$$

$$= -\begin{vmatrix} 0 & -x & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}, \text{ by } C_1 - C_2$$

$$= -x[-xy - y(-x+xy)] = -x[-xy + xy - xy^2] = x^2y^2$$

15. Prove that

$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6.$$

Sol. $$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^3-1 & 3a^2-3 & 3a-3 & 0 \\ a^2-1 & a^2+2a-3 & 2a-2 & 0 \\ a-1 & 2a-2 & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}, \text{ by } R_1 - R_4,\ R_2 - R_4 \text{ and } R_3 - R_4$$

$$= \begin{vmatrix} (a-1)(a^2+a+1) & 3(a-1)(a+1) & 3(a-1) \\ (a-1)(a-1) & (a-1)(a+3) & 2(a-1) \\ (a-1) & 2(a-1) & (a-1) \end{vmatrix}, \text{ expanding by the } C_4$$

$$= (a-1)^3 \begin{vmatrix} a^2+a+1 & 3a+3 & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}, \text{ by taking } (a-1) \text{ common from each row}$$

$$= (a-1)^3 \begin{vmatrix} a^2+a-2 & 3a-3 & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{vmatrix}, \text{ by } R_1 - 3R_3 \text{ and } R_2 - 2R_3$$

$$= (a-1)^3 \begin{vmatrix} (a-1)(a+2) & 3(a-1) \\ (a-1) & (a-1) \end{vmatrix}, \text{ expanding by } C_3$$

$$= (a-1)^5 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix}, \text{ by taking } (a-1) \text{ common from each row}$$

$$= (a-1)^5 (a+2-3) = (a-1)^6$$

16. If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

and $A, B, C, \cdots$ are cofactors of $a, b, c, \cdots$ in Δ, then show that

(i) $BC - F^2 = a\Delta$

Sol. $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(hf - bg)$$

$$= abc - af^2 - ch^2 + ghf + ghf - bg^2$$

$$= abc + 2ghf - af^2 - bg^2 - ch^2$$

$$A = (-1)^2 \begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2, \quad B = (-1)^4 \begin{vmatrix} a & g \\ g & c \end{vmatrix} = ac - g^2,$$

$$C = (-1)^6 \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2, \quad F = (-1)^6 \begin{vmatrix} a & h \\ g & f \end{vmatrix} = gh - af,$$

$$G = (-1)^4 \begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg, \quad H = (-1)^3 \begin{vmatrix} h & f \\ g & c \end{vmatrix} = fg - ch$$

$$BC - F^2 = (ac - g^2)(ab - h^2) - (gh - af)^2$$

$$= a^2bc - ach^2 - abg^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2aghf$$

$$= a^2bc - ach^2 - abg^2 - a^2f^2 + 2aghf$$

$$= a(abc - ch^2 - bg^2 - af^2 + 2ghf) = a\Delta.$$

16.(ii) $CA - G^2 = b\Delta$

Sol. $$CA - G^2 = (ab - h^2)(bc - f^2) - (hf - bg)^2$$

$$= ab^2c - abf^2 - bch^2 + h^2f^2 - h^2f^2 - b^2g^2 + 2bhfg$$

$$= ab^2c - abf^2 - bch^2 - b^2g^2 - 2bhfg$$

$$= b(abc - af^2 - ch^2 - bg^2 + 2ghf) = b\Delta$$

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16.(iii) $AB - H^2 = c\Delta$

Sol. $AB - H^2 = (bc - f^2)(ac - g^2) - (fg - ch)^2$

$= abc^2 - bcg^2 - acf^2 + g^2f^2 - f^2g^2 + 2fgch - c^2h^2$

$= abc^2 - bcg^2 - acf^2 - c^2h^2 + 2fghc$

$= c(abc - bg^2 - af^2 - ch^2 + 2fgh) = c\Delta.$

16.(iv) $GH - AF = f\Delta$

Sol. $GH - AF = (hf - bg)(fg - ch) - (bc - f^2)(gh - af)$

$= hf^2g - cfh^2 - bfg^2 + bgch - bcgh + abcf + ghf^2 - af^3$

$= 2hf^2g - cfh^2 - bfg^2 - af^3 + abcf$

$= f(hfg - ch^2 - bg^2 - af^2 + abc) = f\Delta.$

16.(v) $HF - BG = g\Delta$

Sol. $HF - BG = (gf - ch)(gh - af) - (ac - g^2)(hf - bg)$

$= g^2fh - agf^2 - cgh^2 + achf - achf + abcg + g^2hf - bg^3$

$= 2g^2fh - agf^2 + cgh^2 + abcg - bg^3$

$= g(2hfg - af^2 - ch^2 - bg^2 + abc) = g\Delta.$

16.(vi) $FG - CH = h\Delta$

Sol. $FG - CH = (gh - af)(hf - bg) - (ab - h^2)(fg - ch)$

$= gfh^2 - bhg^2 - af^2h + abfg - abfg + abch + fgh^2 - ch^3$

$= 2gh^2f - bhg^2 - ahf^2 + acbh - ch^3$

$= h(2ghf - bg^2 - af^2 - ch^2 + abc) = h\Delta.$

16.(vii) $aG + hF + gC = 0$

Sol. $aG + hF + gC = a(hf - bg) + h(gh - af) + g(ab - h^2)$

$= ahf - abg + gh^2 - afh + abg - gh^2 = 0$

16.(viii) $hG + bF + fC = 0$

Sol. $hG + bF + fC = h(hf - bg) + b(gh - af) + f(ab - h^2)$

$= h^2f - bgh + bgh - abf + abf - h^2f = 0$

16. (ix) $gG + fF + cC = \Delta.$

Sol. $gG + fF + cC = g(hf - bg) + f(gh - af) + c(ab - h^2)$

$= ghf - bg^2 + ghf - af^2 + abc - ch^2$

$= 2ghf + abc - af^2 - bg^2 - ch^2 = \Delta$

17. If Δ of **Problem 16** is zero, show that

(i) $F^2 + G^2 = C(A + B)$

Sol. Put $\Delta = 0$, in Question 16 (i) and (ii)

$BC - F^2 = a\Delta \Rightarrow BC - F^2 = 0$

$CA - G^2 = b\Delta \Rightarrow CA - G^2 = 0$

These imply that $F^2 = BC$, $G^2 = CA$, so

$F^2 + G^2 = AC + BC = C(A + B).$

17.(ii) $G^2 + H^2 = A(B + C)$

Sol. Put $\Delta = 0$, in **Problem 16 (iii)** and (iv)

$AB - H^2 = c\Delta \Rightarrow AB - H^2 = 0$

$CA - G^2 = b\Delta \Rightarrow CA - G^2 = 0$

These imply that $H^2 = AB$, $G^2 = CA$, so

$G^2 + H^2 = AC + AB = A(B + C).$

17.(iii) $H^2 + F^2 = B(A + C)$

Sol. Put $\Delta = 0$, in **Problem 16 (iii)** and (i)

$AB - H^2 = c\Delta \Rightarrow AB - H^2 = 0$

$BC - F^2 = a\Delta \Rightarrow BC - F^2 = 0$

These imply that $H^2 = AB$, $F^2 = BC$, so

$H^2 + F^2 = AB + BC = B(A + C).$

17.(iv) $ABC = FGH.$

Sol. $F^2G^2H^2 = (BC)(AC)(AB)$ since $F^2 = BC$ etc.

$= A^2B^2C^2$

$\Rightarrow FGH = ABC$

i.e., $ABC = FGH.$

18. Prove that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Sol. $$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ba & c^2+0+a^2 & bc+0+0 \\ 0+ca+0 & cb+0+0 & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

19. Show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2 = 125, \text{ where } \omega \text{ is a fifth root of } 1$$

Sol.
$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix} \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2+\omega^3 & \omega+\omega^3+1+\omega^2 & \omega^2+1+\omega^3+\omega & \omega^3+\omega^2+\omega+1 \\ 1+\omega^2+\omega^4+\omega & \omega+\omega^4+\omega^2+1 & \omega^2+\omega+1+\omega^4 & \omega^3+\omega^3+\omega^3+\omega^3 \\ 1+\omega^3+\omega+\omega^4 & \omega+1+\omega^4+\omega^3 & \omega^2+\omega^2+\omega^2+\omega^2 & \omega^3+\omega^4+1+\omega \\ 1+\omega^4+\omega^3+\omega^2 & \omega+\omega+\omega+\omega & \omega^4+\omega^3+\omega^4+1 & \omega^3+1+\omega^2+\omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} -\omega^4 & -\omega^4 & -\omega^4 & -\omega^4 \\ -\omega^3 & -\omega^3 & -\omega^3 & 4\omega^3 \\ -\omega^2 & -\omega^2 & 4\omega^2 & -\omega^3 \\ -\omega & 4\omega & -\omega & -\omega \end{vmatrix}, \ \omega^5 = 1 \text{ and } 1+\omega+\omega^2+\omega^3+\omega^4 =$$

$$= (-\omega^4)(-\omega^3)(-\omega^2)(-\omega) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 \\ 1 & -4 & 1 & 1 \end{vmatrix}$$

$$= \omega^{10} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -5 \\ 1 & 0 & -5 & 0 \\ 1 & -5 & 0 & 0 \end{vmatrix}, \text{ subtracting the first column from the remaining ones}$$

$$= \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix}, \text{ expanding by the first row}$$

$$= (-5)[0 - 25] = 125$$

20. Prove that the determinant $\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$

is a multiple of the determinant

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and find the other factor.

Sol.
$$\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0a_1+2b_1+c_1 & 3a_1+0b_1+c_1 & 2a_1+3b_1+0c_1 \\ 0a_2+2b_2+c_2 & 3a_2+0b_2+c_2 & 2a_2+3b_2+0c_2 \\ 0a_3+2b_3+c_3 & 3a_3+0b_3+c_3 & 2a_3+3b_3+0c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The other factor is

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -2(0-2) + 1(9-1) = 13.$$

21. Show that the determinant

$$\begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix} \text{ is a multiple of 13.}$$

Sol.
$$\begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 & 2 & 8 & 35282 \\ 4 & 4 & 7 & 5 & 44759 \\ 5 & 8 & 9 & 1 & 58916 \\ 8 & 0 & 6 & 5 & 80652 \\ 9 & 2 & 4 & 6 & 92469 \end{vmatrix}, \text{ by } C_5 + 10\,C_4 + 100\,C_3 + 1000\,C_2 + 10000\,C_1$$

$$= 13\begin{vmatrix} 3 & 5 & 2 & 8 & 2714 \\ 4 & 4 & 7 & 5 & 3443 \\ 5 & 8 & 9 & 1 & 4532 \\ 8 & 0 & 6 & 5 & 6204 \\ 9 & 2 & 4 & 6 & 7113 \end{vmatrix}, \text{ by taking 13 common from } C_5$$

Hence given determinant is a multiple of 13.

22. Prove that

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Sol. $$\Delta = \begin{vmatrix} a^2-2ax+x^2 & a^2-2ay+y^2 & a^2-2az+z^2 \\ b^2-2bx+x^2 & b^2-2by+y^2 & b^2-2bz+z^2 \\ c^2-2cx+x^2 & c^2-2cy+y^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (1)$$

Now $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-b^2 \end{vmatrix} \text{ by } R_2-R_1 \text{ and } R_3-R_1$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

Similarly, $$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

Putting these values into (1), we have

$$\Delta = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

23. Show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

Sol. $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & b^2+ac+ac & -bc+bc-a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

Again, $$\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = -\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= a(bc-a^2) - b(b^2-ac) + c(ab-c^2)$$

$$= -(a^3+b^3+c^2-3abc)$$

So L.H.S. $$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a^3+b^3+c^3-3abc)^2$$

24. Find, by the adjoint method, the inverse of each of the following matrices:

(i) $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol. Let $$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \text{ then}$$

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -3 & -1 & 2 \end{vmatrix}, \text{ by } C_1-2C_3 \text{ and } C_2-C_3$$

$= 1+3 = 4 \neq 0$. Hence A^{-1} exists.

$$\text{Adj } A = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

24.(ii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol. Let $P = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

The cofactors of a, b, etc. are

$$A = (-1)\begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2, \quad B = (-1)^4\begin{vmatrix} a & g \\ g & c \end{vmatrix} = ac - g^2.$$

$$C = (-1)^6\begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2, \quad F = (-1)^5\begin{vmatrix} a & h \\ g & f \end{vmatrix} = gh - af$$

$$G = (-1)^4\begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg, \quad H = (-1)^3\begin{vmatrix} h & f \\ g & c \end{vmatrix} = fg - ch$$

$$\det P = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(hf - bg)$$

$$= abc - af^2 - ch^2 + ghf + ghf - bg^2$$

$$= abc + 2ghf - af^2 - bg^2 - ch^2$$

If $\det P = abc + 2ghf - af^2 - bg^2 - ch \neq 0$, then P^{-1} exists

$$\text{Now} \quad \text{Adj } P = \begin{bmatrix} A & H & G \\ H & B & f \\ G & F & C \end{bmatrix}^T = \begin{bmatrix} A & H & G \\ H & B & f \\ G & F & C \end{bmatrix}$$

$$= \begin{bmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{bmatrix}$$

$$\text{Hence } P^{-1} = \frac{\text{Adj } P}{\det P} = \frac{\begin{bmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{bmatrix}}{abc + 2ghf - af^2 - bg^2 - ch^2}$$

$$= \frac{1}{abc + 2ghf - af^2 - bg^2 - ch^2}\begin{bmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{bmatrix}$$

24.(iii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$, then $\det A = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(-2 + 2) + 1(-4 + 3) + 2(4 - 3)$$

$= 0 - 1 + 2 = 1 \neq 0$. Hence A^{-1} exists.

$$\text{Adj } A = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 2 & -8 & -5 \\ -1 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{\begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}}{1} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

EXERCISE 5.2 (Page 219)

1. Solve for x each of the following equations:

(i) $\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$

Sol. We have

$$0 = \begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2+x & 3 \\ 0 & -3-2x & -3+x \\ 0 & -4-2x & -8 \end{vmatrix}, \text{ by } R_2 - 2R_1, R_3 - 3R_1$$

$$= -8(-3-2x) - (-3+x)(-4-2x), \text{ expanding by } C_1$$

$$= 24 + 16x - (12 + 6x - 4x - 2x^2)$$

$$= 2x^2 + 14x + 12 = 2(x^2 + 7x + 6) = 2(x+6)(x+1)$$

Hence $x = -6, -1$.

1.(ii) $\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$

Sol. Here

$$0 = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1-x^2 & 0 & 0 \\ 2 & 1 & -3 & -1 \\ 2 & 1 & 5 & 3-x^2 \end{vmatrix}, \text{ by } C_2 - C_1, C_3 - 2C_1 \text{ and } C_4 - 3C_1$$

$$= \begin{vmatrix} 1-x^2 & 0 & 0 \\ 1 & -3 & -1 \\ 1 & 5 & 3-x^2 \end{vmatrix}, \text{ expanding by } R_1$$

$$= (1-x^2)[-9 + 3x^2 + 5] = (1-x^2)(3x^2-4)$$

Thus $(1-x^2)(3x^2-4) = 0$

$\Rightarrow \quad 1 - x^2 = 0, \qquad 3x^2 - 4 = 0$

$\Rightarrow \quad x^2 = 1, \qquad x^2 = \frac{4}{3}$

$\Rightarrow \quad x = \pm 1, \qquad x = \pm \frac{2}{\sqrt{3}}$.

1.(iii) $\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$

Sol. We have

$$0 = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 2-x & 2^2-x^2 & 2^3-x^3 \\ 0 & 3-x & 3^2-x^2 & 3^3-x^3 \\ 0 & 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} \text{ by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - R_1$$

$$= \begin{vmatrix} 2-x & 2^2-x^2 & 2^3-x^3 \\ 3-x & 3^2-x^2 & 3^3-x^3 \\ 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix}, \text{ expanding by } C_1$$

$$= (2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 1 & 3+x & 9+3x+x^2 \\ 1 & 4+x & 16+4x+x^2 \end{vmatrix}$$

$$= (2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4 \\ 1 & 3+x & 9 \\ 1 & 4+x & 16 \end{vmatrix}, \text{ by } C_3 - xC_2$$

$$= (2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{vmatrix}, \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= 2(2-x)(3-x)(4-x)$$

Thus $x = 2, 3, 4$.

1.(iv) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1+x & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$

Sol. We have

$$0 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1+x & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & x-1 & 0 & 0 & 0 \\ 0 & 0 & x-2 & 0 & 0 \\ 0 & 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & 0 & x-4 \end{vmatrix}, \quad \text{by } R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1 \text{ and } R_5 - R_1$$

$= (x-1)(x-2)(x-3)(x-4)$ (The det being upper triangular)

Thus $(x-1)(x-2)(x-3)(x-4) = 0$

$\Rightarrow \quad x-1=0, \quad x-2=0, \quad x-3=0, \quad x-4=0$

$\Rightarrow \quad x=1, \quad x=0, \quad x=0, \quad x=0.$

1.(v) $\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$

Sol. Here

$$0 = \begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix}$$

$$= \begin{vmatrix} x+4a & a & a & a & a \\ x+4a & x & a & a & a \\ x+4a & a & x & a & a \\ x+4a & a & a & x & a \\ x+4a & a & a & a & x \end{vmatrix}, \text{ by } C_1 + (C_2 + C_3 + C_4 + C_5)$$

$$= (x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 1 & x & a & a & a \\ 1 & a & x & a & a \\ 1 & a & a & x & a \\ 1 & a & a & a & x \end{vmatrix}, \quad \text{by taking } (x+4a) \text{ common from } C_1$$

$$= (x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 0 & x\ a & 0 & 0 & 0 \\ 0 & 0 & x-a & 0 & 0 \\ 0 & 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & 0 & x-a \end{vmatrix} \quad \text{by } R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1 \text{ and } R_5 - R_1$$

$= (x+4a)(x-a)^4$. (Since the determinant is upper triangular).

Thus $(x+4a)(x-a)^4 = 0$

$\Rightarrow \quad x = a, \quad x = -4a.$

1.(vi) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$

Sol. We have

$$0 = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix},$$

$$= \begin{vmatrix} 4+x & 1 & 1 & 1 \\ 4+x & 1+x & 1 & 1 \\ 4+x & 1 & 1+x & 1 \\ 4+x & 1 & 1 & 1+x \end{vmatrix}, \text{ by } C_1 + (C_2 + C_3 + C_4)$$

$$= (4+x)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & \end{vmatrix}, \quad \text{by taking } (4+x) \text{ common from } C_1$$

$$= (4+x)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x & 0 & 0 \\ 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix}, \quad \text{by } C_2 - C_1,\ C_3 - C_1 \text{ and } C_4 - C_1$$

$= (4+x)x^3.$

Thus $(4+x)x^3 = 0$

$\Rightarrow \quad x = -4, \quad x = 0.$

2. Evaluate each of the following $n \times n$ determinants:

(i) $\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$

$$= \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \cdots & b \\ a+(n-1)b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a+(n-1)b & b & b & \cdots & a \end{vmatrix} \quad \text{by } C_1 + C_i,\ 2 \le i \le n$$

Taking $a+(n-1)b$ common from C_1, we get

$$\Delta = (a+(n-1)b) \begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ 1 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & b & b & \cdots & a \end{vmatrix}$$

$$= [a+(n-1)b] \begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a-b \end{vmatrix}$$

by $R_i - R_1$, $2 \le i \le n$

The last determinant is upper triangular, so its value is equal to the product of its diagonal elements.

So $\Delta = [a+(n-1)b](a-b)^{n-1}$

2.(ii) $\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$

Sol. $\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-n \end{vmatrix}, \quad \text{by } R_1 + (R_2 + R_3 + \cdots + R_n).$$

Since every element of R_1 is zero, so value of the determinant is zero.

2.(iii) $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$$= \begin{vmatrix} \Sigma n & \Sigma n & \Sigma n & \cdots & \Sigma n & \Sigma n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n-2 & n-1 & n & \cdots & n-4 & n-3 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}, \text{ by } R_1 + (R_2 + R_3 + \cdots + R_n)$$

$$= \Sigma n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n-2 & n-1 & n & \cdots & n-4 & n-3 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

$$= \Sigma n \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & n-2 & -1 \\ 3 & 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n-2 & 1 & 2 & \cdots & -2 & -1 \\ n-1 & 1 & 2-n & \cdots & -2 & -1 \\ n & 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix}, \text{ by } C_i - C_1, \; 2 \le i \le n$$

$$= \Sigma n \begin{vmatrix} 1 & 2 & \cdots & n-2 & -1 \\ 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 2 & \cdots & -2 & -1 \\ 1 & 2-n & \cdots & -2 & -1 \\ 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix}, \text{ On expanding by } R_1$$

$$= \Sigma n \begin{vmatrix} 1 & 2 & \cdots & n-2 & -1 \\ 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 2 & \cdots & -2 & -1 \\ 1 & 2-n & \cdots & -2 & -1 \\ -n & -n & \cdots & -2 & -1 \end{vmatrix}, \text{ by } C_1 + C_{n-1};\; C_2 + 2C_{n-1};\; \vdots \;\; C_{n-2} + (n-2)\, C_{n-1}$$

$$= \Sigma n \,(-1)^{n-1} n^{n-2} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

{Taking $-n$ common from the first $(n-2)$ columns and -1 common from the $(n-1)$th column}

{Going through each of the preceding columns $(n-1)$th column shifted to the place of first column, there will be $n-2$ changes of sign. The second last column which now is at the $(n-1)$th position, shifted similarly at the position of second column, there will be $n-3$ changes of sign etc.}

$$= \Sigma n \,(-1)^{n-1} n^{n-2+r} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$= \Sigma n \,(-1)^{n-1+r} n^{n-2} = \Sigma n \,(-1)^{\frac{n(n-1)}{2}} n^{n-2},$$

where $(n-1) + r = (n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \dfrac{n(n-1)}{2}$

2.(iv)
$$\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+2 & \cdots & x \\ \vdots & \vdots & \cdots & \vdots \\ x & x & \cdots & x+n \end{vmatrix}$$

Sol.
$$\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+2 & \cdots & x \\ \vdots & \vdots & \cdots & \vdots \\ x & x & \cdots & x+n \end{vmatrix}$$

$$= \begin{vmatrix} x+1 & x & \cdots & x & x \\ x & x+2 & \cdots & x & x \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x & x & \cdots & x+(n-1) & x \\ x & x & \cdots & x & x+n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & -n \\ 0 & 2 & \cdots & 0 & -n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-1) & -n \\ x & x & \cdots & x & x+n \end{vmatrix} \quad \text{by } R_i - R_n, \; 1 \le i \le n-1$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-1) & 0 \\ x & x & \cdots & x & y \end{vmatrix} \text{ by } C_n + \left[nC_1 + \frac{n}{2}C_2 + \frac{n}{3}C_3 + \cdots + \frac{n}{n-1}C_{n-1}\right]$$

$= 1\cdot2\cdot3 \cdots (n-1)\, y,$ where $y = nx + \frac{n}{2}x + \cdots + \frac{n}{n-1}x + n + x$

$$= nx\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{x} + \frac{1}{n}\right)$$

$$= 1\cdot2\cdot3 \cdots (n-1)\, nx\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{x}\right)$$

$$= xn!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{x}\right)$$

3. If A and B are 3×3 matrices such that $\det(A^2B^3) = 108$ and $\det(A^3B^2) = 72$, find $\det(2A)$ and $\det(B^{-1})$.

Sol. Let $\det A = x$ and $\det B = y$, then

$108 = \det(A^2B^3) = \det A^2 \cdot \det B^3 = (\det A)^2 (\det B)^3 = x^2y^3,$

or $x^2y^3 = 108$ (1)

Also $72 = \det(A^3B^2) = \det A^3 \cdot \det B^2 = (\det A)^3 \cdot (\det B)^2 = x^3y^2$

i.e., $x^3y^2 = 72$ (2)

From equations (1) and (2), we have

$$\frac{(x^3y^2)^3}{(x^2y^3)^2} = \frac{(72)^3}{(108)^2}$$

$$\Rightarrow \quad x^5 = \frac{72\cdot72\cdot72}{108\cdot108} = 32 = 2^5$$

i.e., $\det A = x = 2.$

Substituting the value of x into (1), we have

$4y^3 = 108$ or $y^3 = 27 = 3^3$ or $\det B = y = 3$

In the determinant of the matrix $2A$, we can take 2 common from each row. Since A is of order 3×3, so

$\det 2A = 2^3 \det A$

$= 8x = 8(2) = 16.$

Lastly, $\det B^{-1} = (\det B)^{-1} = (3)^{-1} = \frac{1}{3}.$

4. Let A be an $n \times n$ matrix. Show that

(i) $\det A^m = (\det A)^m$ for any positive integer m

Sol. Since $A^m = A \cdot A \cdots A$, m times, so

$\det A^m = \det(A \cdot A \cdots A)$

$= \det A \cdot \det A \cdots \det A$, m times (by 5.18)

$= (\det A)^m.$

4.(ii) if $\det A^m = 1$, then $\det A = \pm 1$

Sol. Suppose $\det A^m = 1$

$\Rightarrow (\det A)^m = 1$

$\Rightarrow \det A = \pm 1$

For $\det A = 1$, then $(\det A)^m = 1.$

For $\det A = -1$, then $(\det A)^m = 1$, where m is an even integer.

(iii) if $\det A^m = 0$, then $\det A = 0$

Sol. $0 = \det A^m = (\det A)^m$, by **Problem** 4(i)

Thus $(\det A)^m = 0$ implies $\det A = 0$

5. For any nonsingular matrix A, show that

(i) $\det(A^{-1}) = (\det A)^{-1}$

Sol. Since $\det(AB) = \det A \cdot \det B$, so

$\det(AA^{-1}) = \det A \cdot \det(A^{-1})$

$\Rightarrow \det I = \det A \cdot \det(A^{-1})$, also $\det I = 1$

Similarly, $\det I = \det(A^{-1}) \cdot \det A$

Hence $\det A \det(A^{-1}) = \det(A^{-1}) \det A = 1$, showing that

$(\det A)^{-1} = \det(A^{-1}).$

5.(ii) $\det(ABA^{-1}) = \det B$

Sol. $\det(ABA^{-1})$

$= \det A \cdot \det B \cdot \det A^{-1}$

$= \det A \cdot \det(A^{-1}) \cdot \det B$, (Since, the determinant of a matrix is an element of a field and elements of a field commute)

$= \det(A \cdot A^{-1}) \cdot \det A$

$= \det I \cdot \det B = \det B$, as $\det I = 1$

6. For what value of α is the matrix

$$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix} \text{ singular?}$$

Sol. The matrix A is singular if $\det A = 0$

So $\det A = \begin{vmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{vmatrix}$

$= \begin{vmatrix} -\alpha & 3\alpha-1 & 4\alpha+1 \\ 1 & 0 & 0 \\ 2-\alpha & 3\alpha-1 & 4\alpha+1 \end{vmatrix}$, by $C_2 - 2C_1, C_3 - 3C_1$

$= -\begin{vmatrix} 3\alpha-1 & 4\alpha+1 \\ 3\alpha-1 & 4\alpha+1 \end{vmatrix}$, expanding by R_2

$= 0,$ as $R_1 \equiv R_2$

Thus A is singular for all values of α.

7. For

$A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$, verify that $A^{-1} = \dfrac{\text{adj } A}{\det A}$.

Sol. $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$

$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{vmatrix}$

$= -40 + 12 = -28 \neq 0$. Hence A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 1 & 6 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 20 & -2 \\ -2 & -8 & -2 \\ -9 & 6 & 5 \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}$$

Hence $\dfrac{\text{adj } A}{\det A} = \dfrac{-1}{28}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}$

Now $A \cdot \dfrac{\text{adj } A}{\det A} = \dfrac{-1}{28}\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}$

$= \dfrac{-1}{28}\begin{bmatrix} -28 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

Again, $\dfrac{\text{adj } A}{\det A} \cdot A = \dfrac{-1}{28}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$

$= \dfrac{-1}{28}\begin{bmatrix} -28 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

Thus $A \cdot \dfrac{\text{adj } A}{\det A} = \dfrac{\text{adj } A}{\det A} \cdot A = I$, showing that $A^{-1} = \dfrac{\text{adj } A}{\det A}$.

8. Let

$$P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Show that $\det(P_5) = 1$.

Write P_n and evaluate $\det(P_n)$.

Sol. $P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

$\det(P_5) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix}$, by $R_2 - R_1, R_3 - R_1$ $R_4 - R_1$ and $R_5 - R_1$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix}, \text{ by } R_1 + (R_2 + R_3 + R_4 + R_5)$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \text{ expanding by the first row}$$

$$= 1.$$

$$\det(P_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{vmatrix}, \text{ by } R_i - R_1, \ 2 \le i$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{vmatrix}, \text{ by } R_1 + (R_2 + R_3 + \cdots + R_n)$$

$$= (-1)^{n+1} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \vdots & -1 \end{vmatrix}, \text{ expanding by } R_1$$

$$= (-1)^{n+1} (-1)^{n-1} = 1.$$

9. Find the eigenvalues of the given matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = -(1-\lambda^2) - 2 = \lambda^2 - 1$$

$$\lambda^2 - 1 = 0 \quad \Rightarrow \lambda = 1, -1 \text{ are the eigenvalues of } A.$$

9.(ii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Sol. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0 \text{ implies}$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

are the eigenvalues.

9.(iii) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Sol. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)(6 - 5\lambda + \lambda^2 - 1) + \lambda - 3$$

$$= 15 - 20\lambda + 8\lambda^2 - \lambda^3 + \lambda - 3 = -(\lambda^3 - 8\lambda^2 + 19\lambda - 12)$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0 \qquad (1)$$

$\lambda = 1$ is a root of (1).

	1	−8	19	−12
1		1	−7	12
	1	−7	12	0

Other factor of (1) is $\lambda^2 - 7\lambda + 12 = 0$

i.e., $\lambda = 3, 4.$

Eigenvalues are 1, 3, 4.

9.(iv) $\begin{bmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{bmatrix}$

Sol. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 11 \\ 2 & -6-\lambda & 16 \\ 1 & -3 & 7-\lambda \end{vmatrix}$

$= \begin{vmatrix} -\lambda & 0 & 4+\lambda \\ 0 & -\lambda & 2+2\lambda \\ 1 & -3 & 7-\lambda \end{vmatrix}$ by $R_1 - R_3$ and $R_2 - 2R_3$

$= -l(12 - 7l + 6 + 6l) + (4 + l)l$

$= -l3 + l2 - 6l + 4l + l2$

$-(\lambda^3 - 2\lambda^2 + 2\lambda) = 0$ implies

$\lambda(\lambda^2 - 2\lambda + 2) = 0$

$\lambda = 0$ or $\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i.$

Eigenvalues are $\lambda = 0, 1 \pm i$.