Past Paper 2021

Question No: 1

Given a $p \times q$ matrix A with rank r, what is dimensions of column space of A? How can we find out basis of column space of any matrix? Explain with an example.

let we take a matrix A with $p \times q$ order.

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Reduce the matrix into echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - R_1 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-1) \times R_2$$

Now check the columns that has leading elements corresponding to the original matrix of A.

So

$$e_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

So e_1 and e_2 are the basis of column space of matrix A.

We know that rank of matrix is equal to dimension of matrix

So Rank of matrix A is 2

Then

Dimension of A = 2

QUESTION: 2

Find projectile matrix for the coloumn space of a 4x2 matrix A given below

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Projectile of matrix $= A(A^T A)^{-1} A^T$

$$A^T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & -1-1-1-1 \\ -1-1-1-1 & 1+1+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{Adj\,(A^TA)}{|A^TA|}$$

$$Adj\,(A^TA) = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$|A^TA| = \begin{vmatrix} 4 & -4 \\ -4 & 4 \end{vmatrix}$$

$$|A^TA| = 4 \times 4 - (-4) \times (-4)$$
$$= 16 - 16$$
$$= 0$$

$$(A^TA)^{-1} = \frac{\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}}{0}$$

$$(A^TA)^{-1} = 0$$

$A(A^TA)^{-1}A^T = A(0)A^T$
$= 0$
So, Projectile of matrix A is
0.

QUESTION: 3

How can we figure out the given set of vectors, the set is an independent set or not? Explain with an example

Let V be a vector space over field F. The vector space $V_1, V_2, \dots V_n \in V$ are linearly independent set of vectors.
If $a_1V_1 + a_2V_2 + \dots + a_nV_n = 0$
And each a_i's are zero $a_i \in F; i = 1,2,3,\dots$

And if the each a_i's are not zero then given set of vector is linearly dependent set of vectors

let take an example.

We have three set of vectors and determine whether these vectors are linearly dependent or independent.

$V_1 = (3,0,-3)$, $V_2 = (-1,1,2)$ and $V_3 = (4,2,-2)$
$V_4 =$

$a_1 v_1 + b v_2 + c v_3 = 0$

$a(3,0,-3) + b(-1,1,2) + c(4,2,-2) = (0,0,0)$

Where $a, b, c \in F$

$(3a, 0, -3a) + (-b, b, 2b) + (4c, 2c, -2c) = (0,0,0)$

$3a - b + 4c,\ b + 2c,\ -3a + 2b - 2c = (0,0,0)$

$$3a - b + 4c = 0 \quad \text{— (1)}$$
$$0a + b + 2c = 0 \quad \text{— (2)}$$
$$-3a + 2b - 2c = 0 \quad \text{— (3)}$$

Solve eq (1) and (2)

$$\frac{a}{-2-4} = -\frac{b}{6-0} = \frac{c}{3-0} = k$$

$$\frac{a}{-6} = \frac{b}{-6} = \frac{c}{3} = k.$$

$a = -6k$, $b = -6k$ and $c = 3k$.

put values in eq (3)

$$-3(-6k) + 2(-6k) + 2(3k) = 0$$
$$18k - 12k + 6k = 0$$
$$18k - 6k = 0$$
$$12k = 0$$
$$k = 0$$

So $a = b = c = 0$
Then we determine that the given sets of vectors are linearly independent set of vector.

QUESTION: 4

Give an example of 3-D space which is not R^3?