(2020)

Question # 02

(i) Solve the following system of linear equation by Guass Jordan elimination method.

$$2x_1 - x_2 - x_3 = 4$$
$$3x_1 + 4x_2 - 2x_3 = 11$$
$$3x_1 - 2x_2 + 4x_3 = 11$$

Solution:-

$$A_b = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 4 & -2 & 11 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 4 & 11 \end{array}\right] \text{ by } R_{12}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 4 & 11 \end{array}\right] \text{ by } R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & -11 & 1 & -10 \\ 0 & -17 & 7 & -10 \end{array}\right] \text{ by } R_2 - 2R_1 \text{ and } R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & -11 & 1 & -10 \\ 0 & -6 & 6 & 0 \end{array}\right] \text{ by } R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & -11 & 1 & -10 \\ 0 & -1 & 1 & 0 \end{array}\right] \text{ by } 1/6 R_3$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & -10 & 0 & -10 \\ 0 & -1 & 1 & 0 \end{array}\right] \text{ by } R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \text{ by } -1/10 R_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ by } R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ by } R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ by } R_1 - 5R_2$$

$x_1 = 3$

$x_2 = 1$

$x_3 = 1$

(ii) Prove that the following determinant vanishes.

$$\begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Solution:-

$$\text{let } \Delta = \begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc} (0) \quad \because R_1 = R_2$$

$$\Delta = 0$$

$$\text{So, } \begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

(iii) Same as 2017 Q2 (iii)

(iv) Same as 2017 Q2 (iv)

(v) Same as 2018 Q2 (v)

Question # 03.

Same as 2017 Q3.

Question # 04:

Find the eigen values and eigen vectors of

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 7\lambda^2 - 15\lambda + 9 = 0$$

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

Synthetic division:-

$$\begin{array}{c|cccc}
 & 1 & -7 & 15 & -9 \\
1 & \downarrow & 1 & -6 & 9 \\
\hline
 & 1 & -6 & 9 & \big| \; 0
\end{array}$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = 3, 3$$

Eigen values: $\boxed{\lambda = 1, 3, 3}$

For $\lambda = 1$

Eigen vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)\, v = 0$

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce matrix in echelon form $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix} \text{ by } R_{12}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \text{ by } R_1+R_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \text{ by } 1/2\, R_2$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } -2R_2+R_3$$

Now $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x+y-z=0$$

$$y+z=0 \Rightarrow y=-z$$

let $z=a$ be arbitrary value

$$x+(-a)-a=0$$

$$x=2a$$

$$y=-a$$

$$z=a$$

$$v=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix} = a\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Eigen vector $v=[2 \;\; -1 \;\; 1]^T$.

$\lambda = 3$

$(A - \lambda I)V = 0$

$$\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now Reduce matrix in Echelon form $\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \text{ by } R_{12}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ by } 2R_1 + R_2, \; R_1 + R_3$$

Now

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y - z = 0$$

let $y = a$, $z = b$ be arbitrary values.

$$x = a + b$$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix}$$

$$= a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Eigen vector $[1 \quad 1 \quad 0]^T, [1 \quad 0 \quad 1]^T$

Question #05

Reduce the matrix $\begin{bmatrix}6 & 3 & -4\\-4 & 1 & -6\\1 & 2 & -5\end{bmatrix}$

Solution:-

$$\begin{bmatrix}6 & 3 & -4\\-4 & 1 & -6\\1 & 2 & -5\end{bmatrix} R_1 \leftrightarrow R_3$$

$$\begin{bmatrix}1 & 2 & -5\\-4 & 1 & -6\\6 & 3 & -4\end{bmatrix} R_2 + 4R_1$$

$$\begin{bmatrix}1 & 2 & -5\\0 & 9 & -26\\6 & 3 & -4\end{bmatrix} R_3 - 6R_1$$

$$\begin{bmatrix}1 & 2 & -5\\0 & 9 & -26\\0 & -9 & 26\end{bmatrix} 1/9 R_2$$

$$\begin{bmatrix}1 & 2 & -5\\0 & 1 & -26/9\\0 & -9 & 26\end{bmatrix} R_3 + 9R_2$$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix}$$

Question # 6

Determine whether the vectors are linearly independent or not.

$v_1 = (1, -2, 4, 1)$, $v_2 = (2, 1, 0, -3)$,
$v_3 = (1, -6, 1, 4)$

Solution:

let $a(1, -2, 4, 1) + b(2, 1, 0, -3) + c(1, -6, 1, 4) = 0$
where $a, b, c \in F$

$(a, -2a, 4a, a) + (2b, b, 0, -3b) + (c, -6c, c, 4c) = 0$

$(a + 2a + c, -2a + b - 6c, 4a + c, a - 3b + 4c) = 0$

$a + 2b + c = 0 \longrightarrow (1)$

$-2a + b - 6c = 0 \longrightarrow (2)$

$4a + c = 0 \longrightarrow (3)$

$a - 3b + 4c = 0 \longrightarrow (4)$

From (1) and (2)

$$\frac{a}{-12-1} = \frac{-b}{-6+2} = \frac{c}{1+4}$$

$$\frac{a}{-13} = \frac{b}{4} = \frac{c}{5} = k$$

$a = -13k$

$b = 4k$

$c = 5k$

Putting these values in (3) and (4), we see that equations are not satisfied. They are satisfied only when $k = 0$.

$a = b = c = 0$

Hence given vectors in R^4 are linearly independent.

Question # 7

Same as Q6 in 2017.