Exercise No. 5.2

Q1 Solve for x, each of the following equations:

(i) $\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$

Sol. Given

$$\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2+x & 3 \\ 0 & -3-2x & -3+x \\ 0 & -4-2x & -8 \end{vmatrix} = 0 \qquad \text{By } R_2-2R_1,\ R_3-3R_1$$

Expanding from C_1

$$\begin{vmatrix} -3-2x & -3+x \\ -4-2x & -8 \end{vmatrix} = 0$$

$$-8(-3-2x)-(-3+x)(-4-2x) = 0$$

$$24+16x-(12+6x-4x-2x^2) = 0$$

$$24+16x-12-2x+2x^2 = 0$$

$$2x^2+14x+12 = 0$$

$$x^2+7x+6 = 0$$

$$x^2+6x+x+6 = 0$$

$$x(x+6)+1(x+6) = 0$$

$$(x+6)(x+1) = 0$$

$$\Rightarrow \boxed{x = -6, -1}$$

(iii) $$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$$

Sol.

Given $$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1-x^2 & 0 & 0 \\ 2 & 1 & -3 & -1 \\ 2 & 1 & 5 & 3-x^2 \end{vmatrix} = 0 \quad \text{By } C_2-C_1,\ C_3-2C_1,\ C_4-3C_1$$

Expanding from R_1

$$\begin{vmatrix} 1-x^2 & 0 & 0 \\ 1 & -3 & -1 \\ 1 & 5 & 3-x^2 \end{vmatrix} = 0$$

Expanding from R_1

$$(1-x^2)\begin{vmatrix} -3 & -1 \\ 5 & 3-x^2 \end{vmatrix} = 0$$

$$(1-x^2)(-9+3x^2+5) = 0$$

$$(1-x^2)(3x^2-4) = 0$$

$$1-x^2 = 0 \quad , \quad 3x^2-4 = 0$$

$$x^2 = 1 \quad , \quad x^2 = 4/3$$

$$x = \pm 1, \pm \frac{2}{\sqrt{3}}$$

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(iii) $$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

Soln. Given

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 2-x & 2^2-x^2 & 2^3-x^3 \\ 0 & 3-x & 3^2-x^2 & 3^3-x^3 \\ 0 & 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$

$R_2 - R_1$
$R_3 - R_1$
$R_4 - R_1$

Expanding from C_1

$$\begin{vmatrix} 2-x & 2^2-x^2 & 2^3-x^3 \\ 3-x & 3^2-x^2 & 3^3-x^3 \\ 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 1 & 3+x & 9+3x+x^2 \\ 1 & 4+x & 16+4x+x^2 \end{vmatrix} = 0$$

taking $(2-x), (3-x), (4-x)$ common from R_1, R_2, R_3 resp.

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 0 & 1 & 5+x \\ 0 & 2 & 12+2x \end{vmatrix} = 0$$

$R_2 - R_1$
$R_3 - R_1$

Expanding from C_1

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 5+x \\ 2 & 12+2x \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)(12+\cancel{2x}-10-\cancel{2x}) = 0$$

$$(2-x)(3-x)(4-x)(2) = 0$$

$$(2-x)(3-x)(4-x) = 0$$

$$\Rightarrow \quad \boxed{x = 2, 3, 4}$$

(iv)
$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

Sol.-

Given
$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & x-1 & 0 & 0 & 0 \\ 0 & 0 & x-2 & 0 & 0 \\ 0 & 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & 0 & x-4 \end{vmatrix} = 0 \qquad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \\ R_5-R_1 \end{matrix}$$

Expanding from C_1

$$\begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-4 \end{vmatrix} = 0$$

$(x-1)(x-2)(x-3)(x-4) = 0$ ($\because$ det. of a diagonal matrix is equal to the product of diagonal elements.)

$\Rightarrow \boxed{x = 1, 2, 3, 4}$

(v)
$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

Sol.
Given

$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

$$\begin{vmatrix} x+4a & a & a & a & a \\ x+4a & x & a & a & a \\ x+4a & a & x & a & a \\ x+4a & a & a & x & a \\ x+4a & a & a & a & x \end{vmatrix} = 0 \qquad C_1 + (C_2+C_3+C_4+C_5)$$

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 1 & x & a & a & a \\ 1 & a & x & a & a \\ 1 & a & a & x & a \\ 1 & a & a & a & x \end{vmatrix} = 0.$$ taking $x+4a$ common from C_1

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 0 & x-a & 0 & 0 & 0 \\ 0 & 0 & x-a & 0 & 0 \\ 0 & 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

R_2-R_1
R_3-R_1
R_4-R_1
R_5-R_1

Expanding from C_1

$$(x+4a)\begin{vmatrix} x-a & 0 & 0 & 0 \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

$$(x+4a)(x-a)(x-a)(x-a)(x-a) = 0$$

$$\Rightarrow \boxed{x = -4a, a, a, a, a}$$

(vi) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

Sol: Given

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

$$\begin{vmatrix} 4+x & 1 & 1 & 1 \\ 4+x & 1+x & 1 & 1 \\ 4+x & 1 & 1+x & 1 \\ 4+x & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad C_1 + (C_2 + C_3 + C_4)$$

$$(4+x)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad \text{taking } 4+x \text{ common from } C_1$$

$$(4+x).\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = 0 \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

Expanding from C_1

$$(4+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$\Rightarrow \quad (4+x).x^3 = 0$$

$$\Rightarrow$$

$$\boxed{x = -4, 0, 0, 0}$$

Q2 Evaluate each of the following determinants:

(i)
$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$$

$$= \begin{vmatrix} a+(n-1)d & b & b & \cdots & b \\ a+(n-1)d & a & b & \cdots & b \\ a+(n-1)d & b & a & \cdots & b \\ \vdots & & & & \\ a+(n-1)d & b & b & \cdots & b \end{vmatrix} \quad \text{By } C_1+(C_2+C_3+\cdots+C_n)$$

$$= [a+(n-1)d]\begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ 1 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & b & b & \cdots & b \end{vmatrix} \quad \text{taking } a+(n-1)d \text{ common from } C_1$$

$$=[a+(n-1)d]\begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a-b \end{vmatrix} \quad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ \vdots \\ R_n-R_1 \end{matrix}$$

Expanding from C_1

$$\Delta = [a+(n-1)d]\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & a-b \end{vmatrix}$$

$$(a+(n-1)d)(a-b)^{n-1}$$

(ii) $$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

Sol:-

Let $$\Delta = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1-n & 1 & \cdots & 0 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix} \quad R_1+(R_2+R_3+\cdots+R_n)$$

$\Delta = 0 \qquad \because R_1 = 0$

(iii) $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$$= \begin{vmatrix} \Sigma n & \Sigma n & \Sigma n & \cdots & \Sigma n & \Sigma n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & & & & & \\ n-2 & n-1 & n & \cdots & n-4 & n-3 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

Adding R_2 to R_n in R_1 & denoting $1+2+\cdots+n$ by Σn

$$= \Sigma n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-1 & n & & n-4 & n-3 \\ n-1 & n & 1 & & n-3 & n-2 \\ n & 1 & 2 & & n-2 & n-1 \end{vmatrix}$$

taking Σn Common from C_1

$$= \Sigma n \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & n-2 & -1 \\ 3 & 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 1 & 2 & \cdots & -2 & -1 \\ n-1 & 1 & 2-n & \cdots & -2 & -1 \\ n & 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \\ \vdots \\ C_n - C_1 \end{matrix}$$

Expanding from R_1

$$= \Sigma n \begin{vmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & -2 & -1 \\ 1 & 2-n & \cdots & -2 & -1 \\ 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix}$$

$$= \Sigma n \begin{vmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -n & -1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & -n & -1 \\ 0 & -n & \cdots & -2 & -1 \\ -n & -n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_1 + C_{n-1} \\ C_2 + 2C_{n-1} \\ \vdots \\ C_{n-2} + (n-2)C_{n-1} \end{matrix}$$

$$= \Sigma n \cdot (-1)^{n-1} \cdot (n)^{n-2} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

taking $-n$ Common from C_{n-2} & -1 Common from $C_1, C_2, \ldots C_{n-1}$

Going through each of the preceding columns, $(n-1)$th column shifted to the place of first column, there will be $n-2$ changes of sign. The second last column which now is at the $(n-1)$th position shifted similarly at the position of second column, there will be $n-3$ changes of sign etc.

So we have

$$\Delta = \Sigma n (-1)^{n-1} \cdot (n)^{n-2+r} \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\Delta = \Sigma n \cdot (-1)^{(n-1)+r} \cdot (n)^{n-2} \cdot 1$$

Now $(n-1)+\lambda = (n-1)+(n-2)+(n-3)+\cdots+3+2+1$

$$= \frac{(n-1)(n-1+1)}{2}$$

$$= \frac{(n-1)(n)}{2}$$

$$= \frac{n(n-1)}{2}$$

So last eq. becomes

$$\Delta = \Sigma n \cdot (-1)^{\frac{n(n-1)}{2}} \cdot (n)^{n-2}$$

(iv) $\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+2 & \cdots & x \\ \vdots & \vdots & & \vdots \\ x & x & \cdots & x+n \end{vmatrix}$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} x+1 & x & \cdots & x & x \\ x & x+2 & \cdots & x & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & x & \cdots & x+(n-1) & x \\ x & x & \cdots & \cdots & x+n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & -n \\ 0 & 2 & \cdots & 0 & -n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & -n \\ x & x & \cdots & x & x+n \end{vmatrix} \quad \begin{array}{l} R_1 - R_n \\ R_2 - R_n \\ \vdots \\ R_{n-1} - R_n \end{array}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & 0 \\ x & x & \cdots & x & nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) \end{vmatrix} \quad C_n + \left(nC_1 + \frac{n}{2}C_2 + \frac{n}{3}C_3 + \cdots + \frac{n}{n-1}C_{n-1}\right)$$

$$\Delta = 1.2.3 \cdots n-1.\, nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) = n!.\,x.\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right)$$

Q3 If A & B are 3×3 matrices such that $\det(A^2B^3) = 108$ & $\det(A^3B^2) = 72$.
Find $\det(2A)$ & $\det(B^{-1})$.

Sol. Given

$$\left.\begin{aligned} \det(A^2B^3) &= 108 \\ \& \det(A^3B^2) &= 72 \end{aligned}\right]$$

By product theorem

$$\left.\begin{aligned} \det(A^2).\det(B^3) &= 108 \\ \det(A^3).\det(B^2) &= 72 \end{aligned}\right]$$

or

$$(\det A)^2.(\det B)^3 = 108 \quad \text{——— (1)}$$

& $$(\det A)^3.(\det B)^2 = 72 \quad \text{——— (2)}$$

Dividing (1) by (2)

$$\frac{\det B}{\det A} = \frac{108}{72}$$

or $$\frac{\det B}{\det A} = \frac{3}{2}$$

$$\Rightarrow \det B = \frac{3}{2}\det A$$

Put in (2)

$$(\det A)^3.\left(\frac{3}{2}\det A\right)^2 = 72$$

$$\frac{9}{4}(\det A)^5 = 72$$

$$\Rightarrow (\det A)^5 = \frac{72\times 4}{9}$$

$$(\det A)^5 = 32$$

$$\Rightarrow \boxed{\det A = 2}$$

Now $\det(2A) = 2^3.\det A$

$\det(2A) = 8 \times 2$

So

$$\boxed{\det(2A) = 16}$$

Now

$$\det(B^{-1}) = (\det B)^{-1}$$

$$= \left(\tfrac{3}{2}\det A\right)^{-1}$$

$$= \left(\tfrac{3}{2}\times 2\right)^{-1}$$

$$= (3)^{-1}$$

So $\det(B^{-1}) = \frac{1}{3}$

Q4 Let A be an $n\times n$ matrix. Show that

(i) $\det A^m = (\det A)^m$ for any +ve integer m

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol:-

(i) We will prove

$\det A^m = (\det A)^m$ by applying induction on m.

Step ① Let $m = 1$

So $\det A^1 = (\det A)^1$

or $\det A = \det A$

Hence it is true for $m = 1$

Step ② Suppose it is true for $m = K$

i.e., $\det A^K = (\det A)^K$ ——— ① for $K \geq 1$

Step ③ Now we prove it for $m = K+1$

Now

$$\det(A^{k+1}) = \det(A^k . A)$$

$$= (\det A^k).(\det A) \qquad \text{By product theorem}$$

$$= (\det A)^k . (\det A) \qquad \text{using } (1)$$

$$\therefore \det(A^{k+1}) = (\det A)^{k+1}$$

So it is true for $m = k+1$

Hence

$$\det(A^m) = (\det A)^m \quad \text{for all +ve integers } m.$$

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

Sol.-

Since $\det A^m = 1$

So

$$(\det A)^m = 1$$

So $\boxed{\det A = \pm 1}$ where m is an even integer

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol.-

Since $\det A^m = 0$

$$\Rightarrow (\det A)^m = 0$$

$$\Rightarrow \det A = 0$$

Q5 For any non singular matrix C, show that

(i) $\det(C^{-1}) = (\det C)^{-1}$

(ii) $\det(CAC^{-1}) = \det A$

Sol.

(i) Since C is a non singular matrix, so C^{-1} exists such that

$$CC^{-1} = I$$

$$\Rightarrow \det(CC^{-1}) = \det(I)$$

or $\det(C).\det(C^{-1}) = \det(I)$ (By product Theorem)

$$\det C \cdot \det(C^{-1}) = 1$$

$$\det(C^{-1}) = \frac{1}{\det C}$$

$$\det(C^{-1}) = (\det C)^{-1}$$

(ii) $\det(CAC^{-1}) = \det A$

Sol. using product theorem

$$\det(CAC^{-1}) = \det C \cdot \det A \cdot \det C^{-1}$$

$= \det C \cdot \det C^{-1} \cdot \det A$ ∵ det. of an element is an element of field & so they Commute

$= \det(CC^{-1}) \cdot \det A$ By product theorem

$= \det I \cdot \det A$

$= 1 \cdot \det A$

so $\det(CAC^{-1}) = \det A$

Q6 For what value of α is the matrix

$$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$ singular ?

Sol.

Given $$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$

Since A is singular

So $\det A = 0$

$$\Rightarrow \begin{vmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{vmatrix} = 0$$

$$\begin{vmatrix} -\alpha & 3\alpha-1 & 4\alpha+1 \\ 1 & 0 & 0 \\ 2-\alpha & 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0 \qquad \begin{matrix} C_2-2C_1 \\ C_3-3C_1 \end{matrix}$$

Expanding from R_2

$$-\begin{vmatrix} 3\alpha-1 & 4\alpha+1 \\ 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0$$

$$-(3\alpha-1)(4\alpha+1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

obviously matrix A is singular for all values of α.

$$\frac{\text{Adj}\,A}{\det A}\cdot A = \frac{-1}{28}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -2-8-18 & 2-2+0 & -6-12+18 \\ 20-32+12 & -20-8 & 60-48-12 \\ -2-8+10 & 2-2+0 & -6-12-10 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -28 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -28 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\text{Adj}\,A}{\det A}\cdot A = I \qquad \text{———} \quad (2)$$

from (1) & (2)

$$A\cdot\frac{\text{Adj}\,A}{\det A} = \frac{\text{Adj}\,A}{\det A}\cdot A = I$$

$\Rightarrow$

$$A^{-1} = \frac{\text{Adj}\,A}{\det A}$$

Q8 Evaluate

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & -6 \\ 2 & 4 & 4 & -3 \\ 3 & 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} C_3 + C_1 \\ C_4 - 2C_1 \end{matrix}$$

Expanding from R_1

$$= \begin{vmatrix} 3 & 4 & -6 \\ 4 & 4 & -3 \\ 1 & 8 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -20 & 21 \\ 0 & -28 & 33 \\ 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} R_1 - 3R_3 \\ R_2 - 4R_3 \end{matrix}$$

Expanding from C_1

$$= \begin{vmatrix} -20 & 21 \\ -28 & 33 \end{vmatrix}$$

$$= -660 + 588$$

$$\Delta = -72$$