Exercise 6.4

Q1 (i) $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $T(x_1,x_2,x_3) = (x_1,x_2,0)$

Standard basis for $\mathbb{R}^3 = \{e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1)\}$

$T(1,0,0) = (1,0,0) = 1(1,0,0)+0(0,1,0)+0(0,0,1)$

$T(0,1,0) = (0,1,0) = 0(1,0,0)+1(0,1,0)+0(0,0,1)$

$T(0,0,1) = (0,0,0) = 0(1,0,0)+0(0,1,0)+0(0,0,1)$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $T(x_1,x_2,x_3) = (x_1+x_2,\ -x_1-x_2,\ x_3)$

Standard basis for $\mathbb{R}^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

$T(1,0,0) = (1,-1,0) = 1(1,0,0)-1(0,1,0)+0(0,0,1)$

$T(0,1,0) = (1,-1,0) = 1(1,0,0)-1(0,1,0)+0(0,0,1)$

$T(0,0,1) = (0,0,1) = 0(1,0,0)+0(0,1,0)+1(0,0,1)$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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(iii) $$T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (0,-1,0) = 0(1,0,0) - 1(0,1,0) + 0(0,0,1)$$

$$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) - 1(0,0,1)$$

Hence the standard matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(iv) $$T(x_1,x_2,x_3) = (x_1, x_2+x_3, x_1+x_2+x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

$$T(0,1,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

$$T(0,0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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② (i) $T:\mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (3x, 5x)$

Standard basis for $\mathbb{R} = \{1\}$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

$$T(1) = (3,5) = 3(1,0) + 5(0,1)$$

So the matrix of T is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(ii) $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

$$T(1,0,0) = (3,5) = 3(1,0) + 5(0,1)$$

$$T(0,1,0) = (-4,3) = -4(1,0) + 3(0,1)$$

$$T(0,0,1) = (9,-2) = 9(1,0) - 2(0,1)$$

So the matrix of T is $\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$

(iii) $T:\mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2, x_1 + 7x_2, 4x_1).$$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

Standard basis for $\mathbb{R}^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$$T(1,0) = (3,5,1,4) = 3(1,0,0,0) + 5(0,1,0,0) + (0,0,1,0) + 4(0,0,0,1)$$

$$T(0,1) = (4,-2,7,0) = 4(1,0,0,0) - 2(0,1,0,0) + 7(0,0,1,0) + 0(0,0,0,1)$$

So tha matrix of T is $\begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$.

(iv) $T : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$$

Standard basis for $\mathbb{R}^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Standard basis for $\mathbb{R} = \{1\}$

$$T(1,0,0,0) = 2 = 2(1)$$

$$T(0,1,0,0) = 3 = 3(1)$$

$$T(0,0,1,0) = -7 = -7(1)$$

$$T(0,0,0,1) = 1 = 1(1)$$

So the matrix of T is $\begin{bmatrix} 2 & 3 & -7 & 1 \end{bmatrix}$.

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3 Qn (i) Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Since matrix A is of order 3×2, so $m = 3$, $n = 2$.

i.e. $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

or $T(x_1, x_2) = (6x_1 - x_2, x_1 + 2x_2, x_1 + 3x_2)$.

(ii) Let
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

Since matrix A is of order 3×3, so $m=3$, $n=3$.

i.e., $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+2x_3 \\ 2x_1+5x_2+6x_3 \\ -2x_1+3x_2-x_3 \end{bmatrix}$$

or $T(x_1,x_2,x_3) = (x_1+x_2+2x_3,\ 2x_1+5x_2+6x_3,\ -2x_1+3x_2-x_3)$

(iii) Let
$$A = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Since matrix A is of order 3×5, so $m=3$, $n=5$.

i.e., $T:\mathbb{R}^5 \longrightarrow \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_1+x_2+2x_4 \\ x_1+x_4+x_5 \\ -x_2+x_3+x_4+x_5 \end{bmatrix}$$

or

$T(x_1,x_2,x_3,x_4,x_5) = (3x_1+x_2+2x_4,\ x_1+x_4+x_5,\ -x_2+x_3+x_4+x_5)$.

Q4. The matrix of
$$T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

T in terms of coordinates is

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2+x_3 \\ x_1-x_3 \\ -x_1-x_2 \end{bmatrix}$$

or
$$T(x_1,x_2,x_3) = (x_2+x_3,\ x_1-x_3,\ -x_1-x_2)$$

Given basis for $\mathbb{R}^3 = \{v_1=(0,1,2),\ v_2=(1,1,1),\ v_3=(1,0,-2)\}$

$T(v_1) = T(0,1,2) = (3,-2,-1)$

$$(3,-2,-1) = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$(3,-2,-1) = a_1(0,1,2) + a_2(1,1,1) + a_3(3,-2,-1)$$

$$(3,-2,-1) = (a_2 + a_3 \,,\; a_1 + a_2 \,,\; 2a_1 + a_2 - 2a_3)$$

$a_2 + a_3 = 3$, $a_1 + a_2 = -2$, $2a_1 + a_2 - 2a_3 = -1$

$a_2 = 3 - a_3$, $a_1 + (3 - a_3) = -2$

$a_1 = -2 - 3 + a_3$

$a_1 = -5 + a_3$

$2(-5 + a_3) + (3 - a_3) - 2a_3 = -1$

$-10 + 2a_3 + 3 - a_3 - 2a_3 = -1$

$-a_3 - 7 = -1$

$-a_3 = 6$

$\boxed{a_3 = -6}$

$a_2 = 3 - (-6)$

$\boxed{a_2 = 9}$

$a_1 = -5 - 6$

$\boxed{a_1 = -11}$

So $\quad T(v_1) = -11 v_1 + 9 v_2 - 6 v_3 \qquad - \; (1)$

Similarly $\quad T(v_2) = -2 v_1 + 2 v_2 + 0 v_3 \qquad - \; (2)$

and $\quad T(v_3) = 14 v_1 - 11 v_2 + 9 v_3 \qquad - \; (3)$

Hence the matrix of T w.r.t. new basis is

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

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5. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$

Standard basis for $\mathbb{R}^2 = \{(1,0),(0,1)\}$

Standard basis for $\mathbb{R}^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Since $(1,1) = 1(1,0) + 1(0,1)$

and $(-1,1) = -1(1,0) + 1(0,1)$

So $(0,1,2) = T(1,1) = T(1,0) + T(0,1)$ — ①

and $(2,1,0) = T(-1,1) = -T(1,0) + T(0,1)$ — ②

Subtract ② from ①, we get

$$(-2,0,2) = 2T(1,0) + 0$$

$$\Rightarrow \quad T(1,0) = (-1,0,1) = -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Add ① and ② to get

$$(2,2,2) = 2T(0,1)$$

$$\Rightarrow \quad T(0,1) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Q6. $T: M_{23} \to M_{32}$ defined by $T(A) = A^T \in M_{32}$.

Standard basis for $M_{23} = \left\{ \overset{v_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \overset{v_5}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}, \overset{v_6}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$

Standard basis for $M_{32} = \left\{ w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$T(v_1) = w_2 = 0w_1 + w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_2) = w_4 = 0w_1 + 0w_2 + 0w_3 + w_4 + 0w_5 + 0w_6$$

$$T(v_3) = w_6 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + w_6$$

$$T(v_4) = w_1 = w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_5) = w_3 = 0w_1 + 0w_2 + w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_6) = w_5 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + w_5 + 0w_6$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

If we change the order of basis elements then there will be different matrix of T.

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