Vector Spaces

(Exercise 6.1)

Q1: Let V be the set of all infinite sequences in a field F with addition and scalar multiplication defined as below:

For $u = \{a_n\} = a_1, a_2, \ldots, a_n, \ldots \in V$

$v = \{b_n\} = b_1, b_2, \ldots, b_n, \ldots \in V$

$u+v = \{a_n\}+\{b_n\} = a_1+b_1, a_2+b_2, \ldots, a_n+b_n, \ldots$

& $ku = k\{a_n\} = ka_1, ka_2, \ldots, ka_n, \ldots$

where a_n, b_n & k are all in F, $n = 1, 2, 3, \ldots$

Show that V is a vector space over F.

Sol:-

Here $V = \{(a_1, a_2, \ldots, a_n, \ldots) : a_i \in F\}$

First we prove that $(V, +)$ is an abelian group.

(a)
(i) <u>Closure law</u>

Let $v_1 = (a_1, a_2, \ldots)$

& $v_2 = (b_1, b_2, \ldots) \in V$

then $v_1+v_2 = (a_1, a_2, \ldots)+(b_1, b_2, \ldots)$

$= (a_1+b_1, a_2+b_2, \ldots) \in V$

So $v_1, v_2 \in V \Rightarrow v_1+v_2 \in V$

(ii) <u>Associative law</u>

let $v_1 = (a_1, a_2, \ldots)$

$u_2 = (b_1, b_2, \ldots\ldots\ldots)$

$u_3 = (c_1, c_2, \ldots\ldots\ldots) \in V$

then we prove $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

Now $u_1 + (u_2 + u_3) = (a_1, a_2, \ldots\ldots) + [(b_1, b_2, \ldots\ldots) + (c_1, c_2, \ldots\ldots)]$

$= (a_1, a_2, \ldots\ldots) + [(b_1 + c_1, b_2 + c_2, \ldots\ldots\ldots)]$

$= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots\ldots\ldots\ldots]$

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots\ldots\ldots]$

$= (a_1 + b_1, a_2 + b_2, \ldots\ldots) + (c_1, c_2, \ldots\ldots)$

$= [(a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)] + (c_1, c_2, \ldots\ldots)$

$u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

(iii) Identity law

Here $0 = (0, 0, \ldots\ldots\ldots)$ is the additive identity in V because for any $u = (a_1, a_2, \ldots\ldots) \in V$

$u + 0 = (a_1, a_2, \ldots\ldots) + (0, 0, \ldots\ldots)$

$= (a_1 + 0, a_2 + 0, \ldots\ldots\ldots)$

$= (a_1, a_2, \ldots\ldots\ldots)$

$u + 0 = u$

Similarly $0 + u = u$

(iv) Inverse law

Every $u = (a_1, a_2, \ldots\ldots) \in V$ has additive inverse $-u = (-a_1, -a_2, \ldots\ldots) \in V$ because

$u + (-u) = (a_1, a_2, \ldots\ldots) + (-a_1, -a_2, \ldots\ldots)$

$u+(-u) = (a_1-a_1, a_2-a_2, \ldots\ldots)$

$= (0, 0, \ldots\ldots)$

$u+(-u) = 0$

Similarly $-u+u = 0$

(v) <u>Commutative law</u>

Let $u_1 = (a_1, a_2, \ldots\ldots)$

& $u_2 = (b_1, b_2, \ldots\ldots) \in V$

Then we prove $u_1+u_2 = u_2+u_1$

$u_1+u_2 = (a_1, a_2, \ldots\ldots)+(b_1, b_2, \ldots\ldots)$

$= (a_1+b_1, a_2+b_2, \ldots\ldots)$

$= (b_1+a_1, b_2+a_2, \ldots\ldots)$

$= (b_1, b_2, \ldots\ldots)+(a_1, a_2, \ldots\ldots)$

$u_1+u_2 = u_2+u_1$

Hence $(V, +)$ is an abelian group.

(b) <u>Scalar multiplication</u>

(i) Let $a \in F$ & $u = (a_1, a_2, \ldots\ldots) \in V$, then

$au = a(a_1, a_2, \ldots\ldots)$

$= (aa_1, aa_2, \ldots\ldots) \in V$

(ii) Let $a, b \in F$ & $u = (a_1, a_2, \ldots\ldots) \in V$

Then we prove $a(bu) = (ab)u$

Now $a(bu) = a[b(a_1, a_2, \ldots\ldots)]$

$= a[(ba_1, ba_2, \ldots\ldots)]$

$$a(bv) = [a(ba_1), a(ba_2), \dots\dots]$$
$$= [(ab)a_1, (ab)a_2, \dots\dots]$$
$$= (ab)(a_1, a_2, \dots\dots)$$
$$a(bv) = (ab)v$$

(iii) Let $a, b \in F$ & $v = (a_1, a_2, \dots\dots)$

Then we prove $(a+b)v = av + bv$

$$(a+b)v = (a+b)(a_1, a_2, \dots\dots)$$
$$= [(a+b)a_1, (a+b)a_2, \dots\dots]$$
$$= [(aa_1+ba_1), (aa_2+ba_2), \dots\dots]$$
$$= (aa_1, aa_2, \dots\dots) + (ba_1, ba_2, \dots\dots)$$
$$= a(a_1, a_2, \dots\dots) + b(a_1, a_2, \dots\dots)$$
$$(a+b)v = av + bv$$

(iv) Let $a \in F$, $v_1 = (a_1, a_2, \dots\dots)$ & $v_2 = (b_1, b_2, \dots\dots) \in V$

Then we prove $a(v_1+v_2) = av_1 + av_2$

$$a(v_1+v_2) = a[(a_1, a_2, \dots\dots) + (b_1, b_2, \dots\dots)]$$
$$= a[(a_1+b_1, a_2+b_2, \dots\dots)]$$
$$= [a(a_1+b_1), a(a_2+b_2), \dots\dots]$$
$$= [(aa_1+ab_1, aa_2+ab_2, \dots\dots)]$$
$$= (aa_1, aa_2, \dots\dots) + (ab_1, ab_2, \dots\dots)$$
$$= a(a_1, a_2, \dots\dots) + a(b_1, b_2, \dots\dots)$$
$$a(v_1+v_2) = av_1 + av_2$$

(v) let $1 \in F$ & $v = (a_1, a_2, \dots\dots) \in V$

then we prove $1 \cdot v = v$

now $1 \cdot v = 1(a_1, a_2, \ldots)$

$= (1 a_1, 1 a_2, \ldots)$

$= (a_1, a_2, \ldots)$

$1 \cdot v = v$

As all the conditions are satisfied

So V is a vector space over F

Q2 Let V be the set of all ordered pairs of real no.s check whether V is a vector space over R with respect to the indicated operations. If not, state the axioms which fail to hold.

(i) $(a,b)+(c,d) = (a+c, b+d)$

& $k(a,b) = (ka, b)$

Sol.

Here $V = \{(a,b) : a, b \in R\}$

$(a,b)+(c,d) = (a+c, b+d)$

& $k(a,b) = (ka, b)$

First we prove that $(V,+)$ is an abelian group.

(a)

(1) <u>Closure law</u>

Let $v_1 = (a,b)$

& $v_2 = (c,d) \in V$

Then $u_1+u_2 = (a,b)+(c,d)$

$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$ & $u_3 = (e,f) \in V$

Then we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

$u_1+(u_2+u_3) = (a,b)+[(c,d)+(e,f)]$

$= (a,b)+(c+e, d+f)$

$= (a+(c+e), b+(d+f))$

$= ((a+c)+e, (b+d)+f)$

$= (a+c, b+d)+(e,f)$

$= ((a,b)+(c,d))+(e,f)$

$u_1+(u_2+u_3) = (u_1+u_2)+u_3$

(iii) Identity law

Here $0 = (0,0)$ is the additive identity in V because for $u = (a,b) \in V$

$0+u = (0,0)+(a,b)$

$= (0+a, 0+b)$

$= (a,b)$

$0+u = u$

Similarly $u+0 = u$

(iv) Inverse law

Every element $u = (a,b) \in V$ has its additive inverse

$-u = (-a, -b)$ in V because

$$u + (-u) = (a,b) + (-a,-b)$$
$$= (a-a, b-b)$$
$$= (0,0)$$
$$u + (-u) = 0$$

& $$-u + u = (-a,-b) + (a,b)$$
$$= (-a+a, -b+b)$$
$$= (0,0)$$
$$= 0$$

(V) Commutative law

Let $v_1 = (a,b)$, $u_2 = (c,d) \in V$

Then we prove $u_1 + u_2 = u_2 + u_1$

Now $$u_1 + u_2 = (a,b) + (c,d)$$
$$= (a+c, b+d)$$
$$= (c+a, d+b)$$
$$= (c,d) + (a,b)$$
$$u_1 + u_2 = u_2 + u_1$$

Hence $(V,+)$ is an abelian group.

(b) Scalar multiplication

(i) let $a \in R$, $u_1 = (a_1, b_1) \in V$

then $$a u_1 = a(a_1, b_1)$$
$$= (aa_1, ab_1) \in V$$

(ii) Let $a, b \in R$ & $v_1 = (a_1, b_1) \in V$

Then we prove $a(bv_1) = (ab)v_1$

Now $a(bv_1) = a(b(a_1, b_1))$

$= a(ba_1, b_1)$

$= (a(ba_1), b_1)$

$= ((ab)a_1, b_1)$

$= (ab)(a_1, b_1)$

$a(bv_1) = (ab)v_1$

(iii) Let $a, b \in R$, $v_1 = (a_1, b_1) \in V$

Then we prove $(a+b)v_1 = av_1 + bv_1$

Now $(a+b)v_1 = (a+b)(a_1, b_1)$

$= ((a+b)a_1, b_1)$

$= (aa_1 + ba_1, b_1)$

& $av_1 + bv_1 = a(a_1, b_1) + b(a_1, b_1)$

$= (aa_1, b_1) + (ba_1, b_1)$

$= (aa_1 + ba_1, 2b_1)$

So $(a+b)v_1 \neq av_1 + bv_1$

As this Condition is not satisfied

So V is not a vector space over R.

(ii) $(a, b) + (c, d) = (a, b)$

& $K(a, b) = (Ka, Kb)$

Sol. Let $V = \{(a,b) : a,b \in R\}$

Here $(a,b)+(c,d) = (a,b)$

& $k(a,b) = (ka,kb)$

First we prove $(V,+)$ is an abelian group

(u)
(i) Closure law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$

Then $u_1+u_2 = (a,b)+(c,d)$

$= (a,b) \in V$

(ii) Associative law

Let $v_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

Then we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

Now $u_1+(u_2+u_3) = (a,b)+[(c,d)+(e,f)]$

$= (a,b)+(c,d)$

$= (a,b)$

& $(u_1+u_2)+u_3 = [(a,b)+(c,d)]+(e,f)$

$= (a,b)+(e,f)$

$= (a,b)$

So $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

(iii) Identity law

There is no additive identity in V.

As this Condition is not satisfied.

So V is not a vector space over. R.

(iii) $(a,b)+(c,d) = (a+c, b+d)$ & $K(a,b) = (K^2a, K^2b)$

Sol: Let $V = \{(a,b) : a,b \in R\}$

Here $(a,b)+(c,d) = (a+c, b+d)$

& $K(a,b) = (K^2a, K^2b)$

First we prove that $(V,+)$ is an abelian group.

(a) (i) <u>Closure law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$

Then $u_1+u_2 = (a,b)+(c,d)$

$= (a+c, b+d) \in V$

(ii) <u>Associative law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

then we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

Now $u_1+(u_2+u_3) = (a,b)+[(c,d)+(e,f)]$

$= (a,b)+(c+e, d+f)$

$= (a+(c+e), b+(d+f))$

$= ((a+c)+e, (b+d)+f)$

$= (a+c, b+d)+(e,f)$

$= [(a,b)+(c,d)]+(e,f)$

$u_1+(u_2+u_3) = (u_1+u_2)+u_3$

(iii) <u>Identity law</u>

Here $O = (0,0)$ is the additive identity in V because for $u = (a,b) \in V$

$$0+u = (0,0)+(a,b)$$
$$= (0+a, 0+b)$$
$$= (a,b)$$
$$= u$$

$$\& \quad u+0 = (a,b)+(0,0)$$
$$= (a+0, b+0)$$
$$= (a,b)$$
$$= u$$

(iv) Inverse law

Each element $u=(a,b)\in V$ has its additive inverse $-u=(-a,-b)\in V$ such that

$$u+(-u) = (a,b)+(-a,-b) = (a-a, b-b) = (0,0) = 0$$
$$\& \quad -u+u = (-a,-b)+(a,b) = (-a+a, -b+b) = (0,0) = 0$$

(V) Commutative law

Let $u_1=(a,b)$, $u_2=(c,d)\in V$

Then we prove $u_1+u_2 = u_2+u_1$

Now $u_1+u_2 = (a,b)+(c,d)$
$$= (a+c, b+d)$$
$$= (c+a, d+b)$$
$$= (c,d)+(a,b)$$
$$u_1+u_2 = u_2+u_1$$

Hence $(V,+)$ is an abelian group.

(b) Scalar multiplication

(i) Let $\alpha \in R$, $u_1 = (a_1, b_1) \in V$

Then $\alpha u_1 = \alpha(a_1, b_1)$

$= (\alpha^2 a_1, \alpha^2 b_1) \in V$

(ii) Let $a, b \in R$, $u_1 = (a_1, b_1) \in V$

Then we prove $a(bu_1) = (ab)u_1$

Now $a(bu_1) = a\big(b(a_1, b_1)\big)$

$= a(b^2 a_1, b^2 b_1)$

$= \big(a^2(b^2 a_1), a^2(b^2 b_1)\big)$

$= \big((a^2 b^2)a_1, (a^2 b^2)b_1\big)$

$= (ab)(a_1, b_1)$

$a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$, $u_1 = (a_1, b_1) \in V$

Then we prove $(a+b)u_1 = au_1 + bu_1$

Now $(a+b)u_1 = (a+b)(a_1, b_1)$

$= \big((a+b)^2 a_1, (a+b)^2 b_1\big)$

& $au_1 + bu_1 = a(a_1, b_1) + b(a_1, b_1)$

$= (a^2 a_1, a^2 b_1) + (b^2 a_1, b^2 b_1)$

$= (a^2 a_1 + b^2 a_1, a^2 b_1 + b^2 b_1)$

$= \big((a^2 + b^2)a_1, (a^2 + b^2)b_1\big)$

So $(a+b)u_1 \neq au_1 + bu_1$

As this condition is not satisfied

So V is not a vector space over R

(iv) $(a,b)+(c,d) = (a+c, b+d)$ & $k(a,b) = (ka, 0)$

Sol:-

Let $V = \{(a,b) : a, b \in R\}$

Here $(a,b)+(c,d) = (a+c, b+d)$

& $k(a,b) = (ka, 0)$

First we prove that $(V,+)$ is an abelian group.

(a)(i) Closure law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$

then $u_1+u_2 = (a,b)+(c,d)$

$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

Then we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

Now $u_1+(u_2+u_3) = (a,b)+[(c,d)+(e,f)]$

$= (a,b)+(c+e, d+f)$

$= (a+(c+e), b+(d+f))$

$= ((a+c)+e, (b+d)+f)$

$= (a+c, b+d)+(e,f)$

$= [(a,b)+(c,d)]+(e,f)$

$u_1+(u_2+u_3) = (u_1+u_2)+u_3$

(iii) Identity law

Here $O = (0,0)$ is the additive identity in V because for any $u = (a,b) \in V$

$0 + u = (0,0) + (a,b) = (0+a, 0+b) = (a,b) = u$

& $u + 0 = (a,b) + (0,0) = (a+0, b+0) = (a,b) = u$

(iv) Inverse law

Each element $u = (a,b) \in V$ has its additive inverse

$-u = (-a,-b)$ in V because

$u + (-u) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = 0$

& $-u + u = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0) = 0$

(v) Commutative law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$

then we prove $u_1 + u_2 = u_2 + u_1$

Now $u_1 + u_2 = (a,b) + (c,d)$

$= (a+c, b+d)$

$= (c+a, d+b)$

$= (c,d) + (a,b)$

$u_1 + u_2 = u_2 + u_1$

Hence $(V,+)$ is an abelian group.

(b) Scalar multiplication

(i) Let $a \in R$, $u_1 = (a_1, b_1) \in V$

then $a u_1 = a(a_1, b_1)$

$= (a a_1, 0) \in V$

(ii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$

Then we prove $a(bu_1) = (ab)u_1$

Now $a(bu_1) = a(b(a_1, b_1))$

$= a(ba_1, 0)$

$= (a(ba_1), 0)$

$= ((ab)a_1, 0)$

$= (ab)(a_1, b_1)$

$a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$, $u_1 = (a_1, b_1) \in V$

Then we prove $(a+b)u_1 = au_1 + bu_1$

Now $(a+b)u_1 = (a+b)(a_1, b_1)$

$= ((a+b)a_1, 0)$

$= (aa_1 + ba_1, 0)$

$= (aa_1, 0) + (ba_1, 0)$

$= a(a_1, b_1) + b(a_1, b_1)$

$(a+b)u_1 = au_1 + bu_1$

(iv) Let $a \in R$, $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in V$

Then we prove $a(u_1 + u_2) = au_1 + au_2$

Now $a(u_1 + u_2) = a((a_1, b_1) + (a_2, b_2))$

$= a(a_1 + a_2, b_1 + b_2)$

$= (a(a_1 + a_2), 0)$

$= (aa_1 + aa_2, 0)$

$= (aa_1, 0) + (aa_2, 0)$

$\alpha(u_1+u_2) = a(a_1,b_1)+a(a_2,b_2)$

$\alpha(u_1+u_2) = a u_1 + a u_2$

(v) Let $1 \in R$ & $u = (a,b) \in V$

Then we prove $1.u = u$

Now $1.u = 1.(a,b)$

$= (1.a, 0)$

$= (a, 0)$

$\neq (a,b)$

$= u$

So $1.u \neq u$

As this Condition is not satisfied

So V is not a vector space over R.

Q3. Check whether each of the following is a real vector space

(i) The set $C[a,b]$ of all Continuous real valued functions defined on $[a,b]$ with the usual operations on functions as

For $f, g \in C[a,b]$ & $a \in R$

$(f+g)(x) = f(x) + g(x)$

& $(af)(x) = a.f(x)$

Sol:- Let $C[a,b] = \{f : f$ is Continuous real valued fn. defined on $[a,b]\}$

Then clearly $C[a,b]$ is a subset of the vector space V of all real valued continuous functions defined on R

To show that $C[a,b]$ is a vector space over R, we have to show that $C[a,b]$ is a subspace of V

For this, let $f, g \in C[a,b]$

Then both f & g are real valued continuous functions defined on $[a,b]$

Then $(f+g)(x) = f(x) + g(x)$ for $x \in [a,b]$

Now $f+g$ being the sum of two continuous real valued functions defined on $[a,b]$ is also a continuous real valued function defined on $[a,b]$, so $f+g \in C[a,b]$

Hence $f, g \in C[a,b] \Rightarrow f+g \in C[a,b]$

Now let $\alpha \in R$ & $f \in C[a,b]$

Then f is a real valued continuous function defined on $[a,b]$

& $(\alpha f)(x) = \alpha . f(x)$

Clearly scalar multiple of a continuous real valued function is also a continuous real valued function defined on $[a,b]$

So $\alpha \in R$, $f \in C[a,b] \Rightarrow \alpha f \in C[a,b]$

Hence $C[a,b]$ is a subspace of the vector space V of all real valued functions defined on R

Hence $C[a,b]$ is a vector space over R.

(iii) The set of all functions $f \in C[a,b]$ such that

$f(a) = f(b)$

Sol.

Let $\acute{C}[a,b] = \{f : f \in C[a,b]$ & $f(a) = f(b)\}$

Then clearly $\acute{C}[a,b]$ is a subset of $C[a,b]$.

To show that $\acute{C}[a,b]$ is a vector space over R, we have to show that $\acute{C}[a,b]$ is a subspace of the vector space $C[a,b]$.

For this let $f, g \in \acute{C}[a,b]$

Then $f, g \in C[a,b]$ such that

$f(a) = f(b)$

& $g(a) = g(b)$

Now $f+g$ being sum of two real valued continuous functions defined on $[a,b]$ is a real valued continuous function defined on $[a,b]$.

Hence $f+g \in C[a,b]$

Moreover $(f+g)(a) = f(a) + g(a)$

$= f(b) + g(b)$

$(f+g)(a) = (f+g)(b)$

$\Rightarrow f+g \in \acute{C}[a,b]$

So $f,g \in \acute{C}[a,b] \Rightarrow f+g \in \acute{C}[a,b]$

Now let $\alpha \in R$ & $f \in \acute{C}[a,b]$

Then clearly αf is a real valued continuous function defined on $[a,b]$.

So $\alpha f \in C[a,b]$

Moreover

$$(\alpha f)(a) = \alpha \cdot f(a)$$

$$= \alpha \cdot f(b)$$

$$(\alpha f)(a) = (\alpha f)(b) \qquad \because f \in \acute{C}[a,b], \text{ so } f(a) = f(b)$$

Hence $\alpha f \in \acute{C}[a,b]$

So $\alpha \in R$, $f \in \acute{C}[a,b] \Rightarrow \alpha f \in \acute{C}[a,b]$

Hence $\acute{C}[a,b]$ is a subspace of $C[a,b]$.

Thus $C[a,b]$ is a vector space over R.

(iii) The set of all solutions of the diff. eq.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Sol:-

Let W be the set of all solutions of the given diff. equation. Then W is a subset of the vector space V of all real functions defined on R.

To show that W is a vector space over R, we have to show that W is a subspace of V over R.

For this, let $a, b \in R$ & $f, g \in W$

Then f & g are solutions of diff. eq. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Then $\frac{d^2f}{dx^2} - 5\frac{df}{dx} + 6f = 0$
& $\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g = 0$

Now

$$\frac{d^2}{dx^2}(af+bg) - 5\frac{d}{dx}(af+bg) + 6(af+bg)$$
$$= \frac{d^2}{dx^2}(af) + \frac{d^2}{dx^2}(bg) \; 5\frac{d}{dx}(af) - 5\frac{d}{dx}(bg) + 6af + 6bg$$
$$= a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} - 5a\frac{df}{dx} - 5b\frac{dg}{dx} + 6af + 6bg$$
$$= a\left(\frac{d^2f}{dx^2} - 5\frac{df}{dx} + 6f\right) + b\left(\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g\right)$$
$$= a(0) + b(0)$$
$$= 0 + 0$$
$$= 0$$

So $af+bg$ is a solution of diff. eq $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Hence $af+bg \in W$

So $a, b \in R$, $f, g \in W \Rightarrow af+bg \in W$

Hence W is a subspace of V over R.

So W is a vector space over R.

(iv) The set of all 2x2 real matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

Sol:-

Let $V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} : a, b \in R \right\}$

First we prove that V is an abelian group under matrix addition.

This set V has no additive identity.

So V is not a group under addition

As this condition is not satisfied,

So V is not a vector space over R

Q4: Check whether each of the following subsets is a subspace of the indicated vector space:

(i) Q, the set of rational numbers in R.

Sol:

Consider the set Q of rational numbers. Also we know Q is a subset of R

Let $a, b \in R$ & $q_1, q_2 \in Q$

Then q_1 & q_2 are rational numbers.

Since $a, b \in R$ & $q_1, q_2 \in Q$

So $aq_1 + bq_2$ may not be a rational number.

Hence $aq_1 + bq_2 \notin Q$ always

So $a, b \in R$ & $q_1, q_2 \in Q \not\Rightarrow aq_1 + bq_2 \in Q$

Hence Q is not a subspace of R.

(iii) All 2x2 nonsingular real matrices in M_{22}.

Sol. Let V be the set of all nonsingular real matrices in M_{22}. As additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ of V does not belong to V. So V itself is not a vector space over $\mathbb{R}$.

Hence V is not a subspace of M_{22}.

(iii) The set $B[a,b]$ of all bounded real functions defined on $[a,b]$ in the space of all real functions defined on $[a,b]$.

Sol. Here $B[a,b]$ is the set of all bounded real functions defined on $[a,b]$.

Let $a, b \in \mathbb{R}$ & $f, g \in B[a,b]$.

Then f & g are bounded real functions defined on $[a,b]$ then $af + bg$ is also a bounded real valued function defined on $[a,b]$. So $af + bg \in B[a,b]$.

Hence $a, b \in \mathbb{R}$ & $f, g \in B[a,b] \Rightarrow af + bg \in B[a,b]$

So $B[a,b]$ is the subspace of the vector space of all real functions defined on $[a,b]$.

Q5: Show that the union of two subspaces of a vector space need not be a subspace. Let X & Y be subspaces of a vector space V. Prove that $X \cup Y$ is a subspace of V iff either $X \subset Y$ or $Y \subset X$.

Sol: Consider the Euclidean space R^3, where

$R^3 = \{(x,y,z) : x,y,z \in R\}$

Let $X = \{(x_1,0,0) : x_1 \in R\}$

& $Y = \{(0,x_2,0) : x_2 \in R\}$

then clearly X & Y are subspaces of R^3.

we will show that $X \cup Y$ is not a subspace of R^3.

Here $x = (x_1,0,0) \in X$

& $y = (0,x_2,0) \in Y$

then $x, y \in X \cup Y$

But $x+y = (x_1,0,0) + (0,x_2,0)$

$= (x_1,x_2,0) \notin X \cup Y$

As closure property under addition does not hold in $X \cup Y$

So $X \cup Y$ is not a subspace of R^3.

Next.

Suppose $X \cup Y$ is a subspace of V & suppose neither $X \subset Y$ nor $Y \subset X$

Then there are elements x & y such that

$x \in X$ but $x \notin Y$

& $y \in Y$ but $y \notin X$

Now $x, y \in X \cup Y$ and since $X \cup Y$ is a vector space

So $x+y \in X \cup Y$

$\Rightarrow$ either $x+y \in X$ or $x+y \in Y$

Suppose $x+y \in X$

then $y = (x+y) - x \in X$ ($\because X$ is a vector space)

which is contradiction

similarly if $x+y \in Y$

then $x = (x+y) - y \in Y$ ($\because Y$ is a vector space)

which is again contradiction

Hence our supposition is wrong

So either $X \subset Y$ or $Y \subset X$

Conversely

Let $X \subset Y$ or $Y \subset X$

$\Rightarrow X \cup Y = Y$ or $X \cup Y = X$

Since X & Y are subspaces of V

Hence $X \cup Y$ is also a subspace of V.

Q6 Which of the following are subspaces of R^3?

(i) $W = \{(x,y,z) : x+y+z = 0\}$

Sol

Here $W = \{(x,y,z) : x+y+z = 0\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1+y_1+z_1 = 0$ & $x_2+y_2+z_2 = 0$

Now let $a, b \in R$ then

$$aw_1+bw_2 = a(x_1,y_1,z_1)+b(x_2,y_2,z_2)$$
$$= (ax_1,ay_1,az_1)+(bx_2,by_2,bz_2)$$
$$= (ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Here $aw_1+bw_2 \in W$ if $ax_1+bx_2+ay_1+by_2+az_1+bz_2 = 0 \ \forall a,b \in R$

Now

$$ax_1+bx_2+ay_1+by_2+az_1+bz_2 = ax_1+ay_1+az_1+bx_2+by_2+bz_2$$
$$= a(x_1+y_1+z_1)+b(x_2+y_2+z_2)$$
$$= a(0)+b(0)$$
$$= 0+0$$
$$= 0$$

So $aw_1+bw_2 \in W$

Hence for $a,b \in R$ & $w_1,w_2 \in W \Rightarrow aw_1+bw_2 \in W$

So W is a subspace of R^3

(iii) $W = \{(x,y,z) : x \geq 0\}$

Sol:-

Here $W = \{(x,y,z) : x \geq 0\}$

Let $w_1,w_2 \in W$

$\Rightarrow w_1 = (x_1,y_1,z_1)$

& $w_2 = (x_2,y_2,z_2)$ where $x_1,x_2 \geq 0$

Let $a,b \in R$ then

$$aw_1+bw_2 = a(x_1,y_1,z_1)+b(x_2,y_2,z_2)$$

$aw_1 + bw_2 = (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad \forall a, b \in R$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2 \geq 0$

As $a, b \in R$ & $x_1, x_2 \geq 0$

so $ax_1 + bx_2$ may not be ≥ 0

Hence $aw_1 + bw_2 \notin W$ for all $a, b \in R$

So W is not a subspace of R^3

(iii) $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$

Sol:

Here $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$

let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1^2 + y_1^2 + z_1^2 \leq 1$ & $x_2^2 + y_2^2 + z_2^2 \leq 1$

Now let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $(ax_1 + bx_2)^2 + (ay_1 + by_2)^2 + (az_1 + bz_2)^2 \leq 1$

As $(ax_1 + bx_2)^2 + (ay_1 + by_2)^2 + (az_1 + bz_2)^2$

$= a^2(x_1^2 + y_1^2 + z^2 + b^2(x_2^2 + y_2^2 + z_2^2) + 2ab(x_1x_2 + y_1y_2 + z_1z_2)$

If we take $a^2 = \dfrac{1}{x_1^2 + y_1^2 + z_1^2}$ & $b^2 = \dfrac{1}{x_2^2 + y_2^2 + z_2^2}$

and $x_1, y_1, z_1, x_2, y_2, z_2$ are all +ve then above expression is $\neq 1$

So $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

Hence W is not a subspace of R^3.

(IV) $W = \{(x, y, z) : x, y, z \text{ are rationals}\}$

Sol:-

Here $W = \{(x, y, z) : x, y, z \text{ are rationals}\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where x_1, y_1, z_1 & x_2, y_2, z_2 are rationals

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ are rationals.

As $a, b \in R$, so a, b may not be rationals

Hence $ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ may not be rationals

So $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

Hence W is not a subspace of R^3.

(V) $W = \{(x, 0, z) : x, z \in R\}$

Sol:. Here $W = \{(x,0,z) : x, z \in R\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, 0, z_1)$

& $w_2 = (x_2, 0, z_2)$ where $x_1, z_1, x_2, z_2 \in R$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, 0, z_1) + b(x_2, 0, z_2)$

$= (ax_1, 0, az_1) + (bx_2, 0, bz_2)$

$= (ax_1 + bx_2, 0, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, az_1 + bz_2 \in R$

But as $a, b, x_1, z_1, x_2, z_2 \in R$

So $ax_1 + bx_2, az_1 + bz_2 \in R \quad \forall a, b \in R$

Hence $aw_1 + bw_2 \in W$

Thus W is a subspace of R^3.

(vi) $W = \{(x,y,z) : y^2 = x^2 + z^2\}$

Sol:.

Here $W = \{(x,y,z) : y^2 = x^2 + z^2\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $y_1^2 = x_1^2 + z_1^2$ & $y_2^2 = x_2^2 + z_2^2$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$aw_1+bw_2 = (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1+bx_2, ay_1+by_2, az_1+bz_2)$

Now $aw_1+bw_2 \in W$ if $(ay_1+by_2)^2 = (ax_1+bx_2)^2 + (az_1+bz_2)^2$

Here

$(ay_1+by_2)^2 = a^2y_1^2 + b^2y_2^2 + 2aby_1y_2$

$= a^2(x_1^2+z_1^2) + b^2(x_2^2+z_2^2) + 2ab\sqrt{x_1^2+z_1^2}\sqrt{x_2^2+z_2^2}$

$= (a^2x_1^2+b^2x_2^2) + (a^2z_1^2+b^2z_2^2) + 2ab\sqrt{x_1^2+z_1^2}\sqrt{x_2^2+z_2^2}$

$(ay_1+by_2)^2 = (a^2x_1^2+b^2x_2^2) + (a^2z_1^2+b^2z_2^2) + 2ab\sqrt{(x_1^2+z_1^2)(x_2^2+z_2^2)}$

& $(ax_1+bx_2)^2 + (az_1+bz_2)^2 = a^2x_1^2 + b^2x_2^2 + 2abx_1x_2 + a^2z_1^2 + b^2z_2^2 + 2abz_1z_2$

$= (a^2x_1^2+b^2x_2^2) + (a^2z_1^2+b^2z_2^2) + 2ab(x_1x_2+z_1z_2)$

We can see that

$(ay_1+by_2)^2 \neq (ax_1+bx_2)^2 + (az_1+bz_2)^2$

So $aw_1+bw_2 \notin W$ $\quad \forall a,b \in R$

Hence W is not a subspace of R^3.

2018

(vii) $W = \{(x,y,z) : x,y,z \in R,\ 2x+3y-4z = 0\}$

Sol:-

Here $W = \{(x,y,z) : x,y,z \in R,\ 2x+3y-4z = 0\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$, where $2x_1+3y_1-4z_1 = 0$, $2x_2+3y_2-4z_2 =$

Let $a, b \in R$ then

$$aw_1+bw_2 = a(x_1,y_1,z_1)+b(x_2,y_2,z_2)$$
$$=(ax_1,ay_1,az_1)+(bx_2,by_2,bz_2)$$
$$=(ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Now $aw_1+bw_2 \in W$ if $2(ax_1+bx_2)+3(ay_1+by_2)-4(az_1+bz_2)=0$

Now $2(ax_1+bx_2)+3(ay_1+by_2)-4(az_1+bz_2)$

$$=2ax_1+2bx_2+3ay_1+3by_2-4az_1-4bz_2$$
$$=(2ax_1+3ay_1-4az_1)+(2bx_2+3by_2-4bz_2)$$
$$=a(2x_1+3y_1-4z_1)+b(2x_2+3y_2-4z_2)$$
$$=a(0)+b(0)$$
$$=0+0$$
$$=0$$

Hence $2(ax_1+bx_2)+3(ay_1+by_2)-4(az_1+bz_2)=0$

So $aw_1+bw_2 \in W$

Thus W is a subspace of R^3

Q7: Let V be the vector space of all real valued functions defined on R. State which of the following are subspaces of V

(i) The set of all even functions

Sol:

Here $V=\{f : f$ is a real valued function defined on $R\}$

& $W=\{f : f$ is an even function$\}$

Let $f, g \in W$

then both f & g are even functions

i.e, $f(-x) = f(x)$

& $g(-x) = g(x)$

Let $a, b \in R$, then we prove $af+bg \in W$

Now $(af+bg)(-x) = (af)(-x) + (bg)(-x)$

$= a.f(-x) + b.g(-x)$

$= a.f(x) + b.g(x)$

$= (af)(x) + (bg)(x)$

$= (af+bg)(x)$

So $af+bg$ is an even function, hence $af+bg \in W$

Thus $a, b \in R$ & $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V

(ii) $W = \{f : f$ is a differentiable function$\}$

Sol:

Here $W = \{f : f$ is a diff function$\}$

Let $f, g \in W$

Then both f & g are differentiable functions

i.e., f' & g' exists

Now let $a, b \in R$ then we prove $af+bg \in W$

As $(af+bg)' = (af)' + (bg)'$

$(af+bg)' = af' + bg'$

Since f', g' exist, so $(af+bg)'$ exists.

Hence $af+bg$ is differentiable, so $af+bg \in W$

Hence $a, b \in R$ & $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V.

(iii) $W = \{f : f(x) = Kf(-x),\ K \in R \text{ fixed}\}$

Sol:

Here $W = \{f : f(x) = Kf(-x),\ K \in R \text{ fixed}\}$

Let $f, g \in W$

$\Rightarrow f(x) = Kf(-x)$

& $g(x) = Kg(-x)$, where $K \in R$ is fixed.

Let $a, b \in R$ then we prove $af+bg \in W$

Now

$$
\begin{aligned}
(af+bg)(x) &= (af)(x) + (bg)(x) \\
&= a.f(x) + b.g(x) \\
&= a.Kf(-x) + b.Kg(-x) \\
&= K.a.f(-x) + K.b.g(-x) \qquad (\because a, b, K \in R) \\
&= K[a.f(-x) + b.g(-x)] \\
&= K[(af)(-x) + (bg)(-x)] \\
&= K(af+bg)(-x)
\end{aligned}
$$

So $af+bg \in W$.

Hence $a, b \in R$ & $f, g \in W \Rightarrow af + bg \in W$

So W is a subspace of V.

(iv) $W = \{ f \in V : \int_0^1 f(x)dx = 0 \}$

Sol:-

Here $W = \{ f \in V : \int_0^1 f(x)dx = 0 \}$

Let $f, g \in W$

$\Rightarrow \int_0^1 f(x)dx = 0$ & $\int_0^1 g(x)dx = 0$

Let $a, b \in R$ then we show that $af + bg \in W$

As $\int_0^1 (af+bg)(x)dx = \int_0^1 [(af)(x) + (bg)(x)]\,dx$

$$= \int_0^1 [a.f(x) + b.g(x)]\,dx$$

$$= \int_0^1 a.f(x)dx + \int_0^1 b.g(x)\,dx$$

$$= a\int_0^1 f(x)dx + b\int_0^1 g(x)dx$$

$$= a(0) + b(0)$$

$$= 0 + 0$$

$$= 0$$

So $af + bg \in W$

Hence $a, b \in R$ & $f, g \in W \Rightarrow af + bg \in W$

So W is a subspace of V.

Q8: Let V be the vector space of all real polynomials of degree $\leq n$ together with the zero polynomial. Determine whether or not W is a subspace of V, where W consists of the zero polynomial and all polynomials

(i) with integral coefficients and of degree $\leq n$.

Sol:.

Here $V = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_i\text{'s} \in R\}$

& $W = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_i\text{'s} \in Z\}$

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

& $w_2 = b_0 + b_1 x + b_2 x^2 + \dots + b_m x^m$ where $m < n$, $a_i, b_i \in Z$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(a_0 + a_1 x + \dots + a_n x^n) + b(b_0 + b_1 x + \dots + b_m x^m)$$

$$= aa_0 + aa_1 x + \dots + aa_n x^n + bb_0 + bb_1 x + \dots + bb_m x^m$$

$$= (aa_0 + bb_0) + (aa_1 + bb_1)x + \dots + (aa_m + bb_m)x^m + \dots + aa_n x^n$$

As $a, b \in R$ & $a_i, b_i \in Z$

So $aa_i + bb_i$ may not be integers, $1 \leq i \leq n$

Hence $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

So W is not a subspace of V

(ii)

Sol:- Here $W = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : n \leq 3\}$

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1x + a_2x^2 + a_3x^3$

& $w_2 = b_0 + b_1x + b_2x^2$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(a_0 + a_1x + a_2x^2 + a_3x^3) + b(b_0 + b_1x + b_2x^2)$

$= aa_0 + aa_1x + aa_2x^2 + aa_3x^3 + bb_0 + bb_1x + bb_2x^2$

$= (aa_0 + bb_0) + (aa_1 + bb_1)x + (aa_2 + bb_2)x^2 + aa_3x^3$

which is a polynomial of degree ≤ 3

Hence $aw_1 + bw_2 \in W$

So $a, b \in R$ & $w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W$

Hence W is a subspace of V.

(iii) with only even powers of x & of degree $\leq n$

Sol:-

Here $W = \{a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} : n \in Z^+\}$

Let $w_1, w_2 \in W$.

$\Rightarrow w_1 = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$

& $w_2 = b_0 + b_1x^2 + b_2x^4 + \cdots + b_mx^{2m}$, where $m < n$, $m, n \in Z^+$

Let $a, b \in R$ Then

$aw_1 + bw_2 = a(a_0 + a_1x^2 + a_2x^2 + \cdots + a_nx^{2n}) + b(b_0 + b_1x^2 + b_2x^4 + \cdots + b_mx^{2m})$

$= aa_0 + aa_1x^2 + aa_2x^2 + \cdots + aa_nx^{2n} + bb_0 + bb_1x^2 + bb_2x^4 + \cdots + bb_mx^{2m}$

$$= (aa_0 + bb_0) + (aa_1 + bb_1)x^2 + (aa_2 + bb_2)x^4 + \dots + (aa_m + bb_m)x^{2m} + \dots + aa_n x^{2n}$$

which is a polynomial with only even powers of x

So $aw_1 + bw_2 \in W$

Hence W is a subspace of V.

2016

Q9: Express the vector $(2,-5,3)$ as a linear combination of the vectors $(1,-3,2)$, $(2,-4,-1)$ & $(1,-5,7)$.

Sol: For $a,b,c \in R$

Let $(2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$

$$= (a,-3a,2a) + (2b,-4b,-b) + (c,-5c,7c)$$

or $(2,-5,3) = (a+2b+c, -3a-4b-5c, 2a-b+7c)$

$\Rightarrow$ $a + 2b + c = 2$ ——— (1)

$-3a - 4b - 5c = -5$ ——— (2)

$2a - b + 7c = 3$ ——— (3)

Multiplying (1) & 3 & adding in (2)

$$\cancel{3a} + 6b + 3c = 6$$
$$-\cancel{3a} - 4b - 5c = -5$$
$$2b - 2c = 1$$

$b - c = 1/2$ ——— (4)

Multiplying (1) by 2 & subt. (3) from (1)

$$\cancel{2a} + 4b + 2c = 4$$
$$-\cancel{2a} \mp b \pm 7c = -3$$
$$5b - 5c = 1$$

$b - c = \frac{1}{5}$ ——— (5)

From (4) & (5) we cannot find values of b & c.

Thus $(2,-5,3)$ cannot be expressed as a linear combination of $(1,-3,2), (2,-4,-1)$ & $(1,-5,7)$.

Q10: For what value of K will the vector $(1,-2,K)$ in R^3 be a linear combination of the vectors $(3,0,-2)$ & $(2,-1,-5)$?

Sol:- For $a, b \in R$

Let $(1,-2,K) = a(3,0,-2) + b(2,-1,-5)$

$= (3a, 0, -2a) + (2b, -b, -5b)$

or $(1,-2,K) = (3a+2b, -b, -2a-5b)$

$\Rightarrow \quad 3a + 2b = 1$ ——— (1)

$-b = -2$ ——— (2)

& $-2a - 5b = K$ ——— (3)

from (2) $\boxed{b = 2}$

Put in (1)

$3a + 2(2) = 1$

$3a + 4 = 1$

$3a = 1 - 4$

$3a = -3$

$\boxed{a = -1}$

Put values in (3)

$2(-1)-5(2) = K$

$2-10 = K$

$-8 = K$

$\Rightarrow \boxed{K=-8}$

So for $K=-8$, the vector $(1,-2,K)$ is a linear combination of $(3,0,-2)$ & $(2,-1,-5)$.

Q11: Let U & W be the subspaces of R^3 defined by

$U = \{(x,y,z) : x=y=z\}$

$W = \{(0,y,z) : y,z \in R\}$

Show that $R^3 = U \oplus W$

Sol:-

To show that $R^3 = U \oplus W$, we have to prove that

$R^3 = U+W$

& $U \cap W = \{0\}$

Here $U = \{(x,y,z) : x=y=z\}$

& $W = \{(0,y,z) : y,z \in R\}$

Let $(x,y,z) \in R^3$

Then $(x,y,z) = (x,x,x)+(0,y-x,z-x) \in U+W$

So $(x,y,z) \in U+W$

Hence $R^3 \subseteq U+W$ ——— ①

Conversely

let $u \in U$ & $w \in W$ so that $u+w \in U+W$

$\Rightarrow u = (x, x, x)$ & $w = (0, y, z)$

then $u + w = (x, x, x) + (0, y, z)$

$= (x, x+y, x+z) \in R^3$

so $u + w \in R^3$

Hence $U + W \subseteq R^3$ —— (2)

from (1) & (2)

$R^3 = U + W$

Next, we want to show that $U \cap W = \{0\}$

Let $\alpha \in U \cap W$

$\Rightarrow \alpha \in U$ & $\alpha \in W$

$\Rightarrow \alpha = (x, x, x)$ & $\alpha = (0, y, z)$

So $(x, x, x) = (0, y, z)$

$\Rightarrow x = 0, \quad x = y, \quad x = z$

So $x = y = z = 0$

Hence $\alpha = (0, 0, 0)$

So $U \cap W = \{0\}$

Hence $R^3 = U \oplus W$

Q12: Show that each of the following sets of vectors generates R^3

(i) $(1,2,3), (0,1,2), (0,0,1)$

Sol:-

Given set is $\{(1,2,3),(0,1,2),(0,0,1)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,2,3) + b(0,1,2) + c(0,0,1)$ for $a,b,c \in R$

$= (a,2a,3a) + (0,b,2b) + (0,0,c)$

$(x,y,z) = (a, 2a+b, 3a+2b+c)$

$\Rightarrow \quad a = x$ ——— ①

$2a+b = y$ ——— ②

$3a+2b+c = z$ ——— ③

from ① $\boxed{a = x}$

Put in ②

$2x+b = y$

$\boxed{b = y-2x}$

Put values of a & b in ③

$3x + 2(y-2x) + c = z$

$3x + 2y - 4x + c = z$

$-x + 2y + c = z$

$\boxed{c = x-2y+z}$

So

$(x,y,z) = x(1,2,3) + (y-2x)(0,1,2) + (x-2y+z)(0,0,1)$

Hence given vectors generate R^3.

(ii) $(1,1,1), (0,1,1), (0,1,-1)$

Sol:- Given set is $\{(1,1,1),(0,1,1),(0,1,-1)\}$

Let $(x,y,z) \in R^3$ & suppose

$$(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,1,-1)$$

$$= (a,a,a) + (0,b,b) + (0,c,-c) \quad \text{for } a,b,c \in R$$

$$(x,y,z) = (a, a+b+c, a+b-c)$$

$$\Rightarrow \quad a = x \quad \text{———} \; (1)$$

$$a+b+c = y \quad \text{———} \; (2)$$

$$a+b-c = z \quad \text{———} \; (3)$$

from (1) $\boxed{a = x}$

Adding (2) & (3)

$$2a + 2b = y + z$$

$$2x + 2b = y + z$$

$$2b = y + z - 2x$$

$$\boxed{b = \frac{y+z-2x}{2}}$$

Put values of a & b in (2)

$$x + \frac{y+z-2x}{2} + c = y$$

$$c = y - x - \frac{y+z-2x}{2}$$

$$= \frac{2y-2x-y-z+2x}{2}$$

$$\boxed{c = \frac{y-z}{2}}$$

So

$$(x,y,z) = x(1,1,1) + \left(\frac{y+z-2x}{2}\right)(0,1,1) + \left(\frac{y-z}{2}\right)(0,1,-1)$$

Hence given vectors generate R^3.

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Q13: Determine whether the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ spans R^3.

Sol:

Given set is $S = \{(1,1,2),(1,0,1),(2,1,3)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,1,2) + b(1,0,1) + c(2,1,3)$ for $a,b,c \in R$

$= (a,a,2a) + (b,0,b) + (2c,c,3c)$

$(x,y,z) = (a+b+2c,\ a+c,\ 2a+b+3c)$

$\Rightarrow \quad a+b+2c = x$ ——— ①

$a+c = y$ ——— ②

$2a+b+3c = z$ ——— ③

Subt. ③ from ①

$-a-c = x-z$

or $\quad a+c = z-x$ ——— ④

& $\quad a+c = y$ ——— ②

The equations ② & ④ cannot be solved for a & c.

Hence we cannot find values of a, b & c

So the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ does not span R^3.

Q14: Show that the yz-plane $W = \{(0,y,z) : y,z \in R\}$ is spanned by

(ii) $(0,1,1)$ and $(0,2,-1)$

Sol:-

Suppose $W = \{(0,y,z) : y,z \in R\}$

Let $(0,y,z) \in W$ &

$(0,y,z) = a(0,1,1) + b(0,2,-1)$ for $a, b \in R$

$= (0,a,a) + (0,2b,-b)$

$(0,y,z) = (0, a+2b, a-b)$

$\Rightarrow$ $a+2b = y$ —— ①

& $a-b = z$ —— ②

Subt. ② from ①

$3b = y-z$

$$\boxed{b = \frac{y-z}{3}}$$

Put value in ②

$$a - \frac{y-z}{3} = z$$

$$a = \frac{y-z}{3} + z$$

$$= \frac{y-z+3z}{3}$$

$$\boxed{a = \frac{y+2z}{3}}$$

So

$$(0,y,z) = \left(\frac{y+2z}{3}\right)(0,1,1) + \left(\frac{y-z}{3}\right)(0,2,-1)$$

Hence yz-plane is spanned by $(0,1,1)$ & $(0,2,-1)$

(iii) $(0,1,2), (0,2,3), (0,3,1)$

Sol:. Suppose $W = \{(0,1,2),(0,2,3),(0,3,1)\}$

Let $(0,y,z) \in W$ &

$(0,y,z) = a(0,1,2) + b(0,2,3) + c(0,3,1)$ for $a,b,c \in R$

$= (0,a,2a) + (0,2b,3b) + (0,3c,c)$

$(0,y,z) = (0, a+2b+3c, 2a+3b+c)$

$\Rightarrow \quad a+2b+3c = y$ ——— (1)

& $\quad 2a+3b+c = z$ ——— (2)

Put $a=0$ in (1) & (2)

$2b+3c = y$ ——— (3)

$3b+c = z$ ——— (4)

Multiplying (4) by 3 & subt. from (3)

$2b + \cancel{3c} = y$ ——— (3)

$-9b \pm \cancel{3c} = -3z$ ——— (5)

$-7b = y-3z$

$7b = 3z-y$

$\boxed{b = \frac{3z-y}{7}}$

Put in (3)

$2\left(\frac{3z-y}{7}\right) + 3c = y$

$\frac{6z-2y}{7} + 3c = y$

$3c = y - \frac{6z-2y}{7}$

$= \frac{7y-6z+2y}{7}$

$3c = \frac{9y-6z}{7}$

$$\boxed{c = \frac{3y-2z}{7}}$$

Hence

$$(0,y,z) = c(0,1,2) + \left(\frac{3z-y}{7}\right)(0,2,3) + \left(\frac{3y-2z}{7}\right)(0,3,1)$$

So yz-plane is spanned by $(0,1,2)$, $(0,2,3)$ & $(0,3,1)$

Q15: Find an equation (or equations) of the subspace W of R^3 generated by each of the following sets of vectors.

(i) $\{(1,-3,5), (-2,6,-10)\}$

Sol:-

Since W is spanned by the vectors $(1,-3,5)$ & $(-2,6,-10)$ so each vector $(x,y,z) \in W$ is a linear combination of these vectors. i.e., there exist scalars a, b such that

$$(x,y,z) = a(1,-3,5) + b(-2,6,-10)$$
$$= (a,-3a,5a) + (-2b,6b,-10b)$$
$$(x,y,z) = (a-2b, -3a+6b, 5a-10b)$$

$$\Rightarrow \left.\begin{aligned} a-2b &= x \\ -3a+6b &= y \\ 5a-10b &= z \end{aligned}\right\} \text{--- (1)}$$

We reduce the augmented matrix of this system to echelon form as.

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 6 & y \\ 5 & -10 & z \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & 0 & y+3x \\ 0 & 0 & z-5x \end{array}\right] \quad \begin{array}{l} R_2+3R_1 \\ R_3-R_1 \end{array}$$

The above system ① is consistent if

rank A = rank A_b

$\Rightarrow$ $y+3x=0$

& $z-5x=0$

or $y=-3x$

$z=5x$

or $\left.\begin{array}{l} x=t \\ y=-3t \\ z=5t \end{array}\right\} \quad t\in R$

These are the required equations of the subspace W of R^3.

(ii) $\{(1,-3,2),(-2,0,3)\}$

Sol.

Since W is spanned by the vectors $(1,-3,2)$ & $(-2,0,3)$

So each vector $(x,y,z)\in W$ is a linear combination of these vectors.

i.e, there exist scalars $a,b\in R$ such that

$(x,y,z) = a(1,-3,2)+b(-2,0,3)$

$\qquad = (a,-3a,2a)+(-2b,0,3b)$

$(x,y,z) = (a-2b,-3a,2a+3b)$

$\Rightarrow \left.\begin{array}{l} a-2b = x \\ -3a = y \\ 2a+3b = z \end{array}\right\}$ ——— ①

we reduce the augmented matrix of system (1) to echelon form as:

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 0 & y \\ 2 & 3 & z \end{bmatrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & x \\ 0 & -6 & y+3x \\ 0 & 7 & z-2x \end{bmatrix} \quad \begin{matrix} R_2+3R_1 \\ R_3-2R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & x \\ 0 & 1 & -\frac{y+3x}{6} \\ 0 & 1 & \frac{z-2x}{7} \end{bmatrix} \quad -\frac{1}{6}R_2 , \frac{1}{7}R_3$$

$$\begin{bmatrix} 1 & -2 & x \\ 0 & 1 & -\frac{y+3x}{6} \\ 0 & 0 & \frac{z-2x}{7} + \frac{y+3x}{6} \end{bmatrix} \quad R_3 - 7R_2$$

The system (1) is consistent if

$$\text{rank } A = \text{rank } A_b$$

$$\Rightarrow \frac{z-2x}{7} + \frac{y+3x}{6} = 0$$

Multiplying both sides by 42

$$6(z-2x) + 7(y+3x) = 0$$

$$6z - 12x + 7y + 21x = 0$$

$$9x + 7y + 6z = 0$$

which is the required equation of the subspace W of R^3

(iii) $\{(1,-2,1), (-2,0,3), (3,-2,-2)\}$

Sol:

Since W is spanned by the vectors $(1,-2,1)$, $(-2,0,3)$ & $(3,-2,-2)$. So each vector $(x,y,z) \in W$ is a linear combination of these vectors.

i.e., there exist scalars $a,b,c \in R$ such that

$$(x,y,z) = a(1,-2,1) + b(-2,0,3) + c(3,-2,-2)$$

$$= (a,-2a,a) + (-2b,0,3b) + (3c,-2c,-2c)$$

$$(x,y,z) = (a-2b+3c,\ -2a-2c,\ a+3b-2c)$$

$$\Rightarrow \left.\begin{aligned} a-2b+3c &= x \\ -2a- \quad &= y \\ a+3b-2c &= z \end{aligned}\right\} \text{———} \ ①$$

We reduce the augmented matrix of this system to echelon form as

$$A_b = \begin{bmatrix} 1 & -2 & 3 & x \\ -2 & 0 & -2 & y \\ 1 & 3 & -2 & z \end{bmatrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & 3 & x \\ 0 & -4 & 4 & y+2x \\ 0 & 5 & -5 & z-x \end{bmatrix} \quad \begin{matrix} R_2+2R_1 \\ R_3-R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & 3 & x \\ 0 & 1 & -1 & -\frac{1}{4}(y+2x) \\ 0 & 1 & -1 & \frac{1}{5}(z-x) \end{bmatrix} \quad -\frac{1}{4}R_2,\ \frac{1}{5}R_3$$

$$\overset{R}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 3 & x \\ 0 & 1 & -1 & -\frac{1}{4}(y+2x) \\ 0 & 0 & 0 & \frac{1}{5}(z-x)+\frac{1}{4}(y+2x) \end{array}\right] \quad R_3-R_2$$

The system ① is Consistent if

rank A = rank A_b

$\Rightarrow \frac{1}{5}(z-x) + \frac{1}{4}(y+2x) = 0$

Multiplying both sides by 20

$4(z-x) + 5(y+2x) = 0$

$4z - 4x + 5y + 10x = 0$

$6x + 5y + 4z = 0$

which is the required equation of the subspace W of R^3.

Q16: Show that the Complex numbers $2+3i$ & $1-2i$ generate the vector space C over R.

Sol:-

Here $C = \{x+iy : x, y \in R\}$

Any vector of C has the form $x+iy$; $x, y \in R$

Suppose

$$x+iy = a(2+3i) + b(1-2i)$$

$$= 2a + 3ai + b - 2bi$$

$$x+iy = (2a+b) + i(3a-2b)$$

Comparing real & imaginary parts.

$2a + b = x$ ——— ①

$3a - 2b = y$ ——— ②

Multiplying ① by 2 & adding in ②

$$4a + 2b = 2x \quad \text{—— (1)}$$

$$3a - 2b = y \quad \text{—— (2)}$$

$$7a = 2x + y$$

$$\boxed{a = \frac{2x+y}{7}}$$

Put in (1)

$$2\left(\frac{2x+y}{7}\right) + b = x$$

$$\frac{4x+2y}{7} + b = x$$

$$b = x - \frac{4x+2y}{7}$$

$$= \frac{7x-4x-2y}{7}$$

$$\boxed{b = \frac{3x-2y}{7}}$$

Hence

$$x + iy = \left(\frac{2x+y}{7}\right)(2+3i) + \left(\frac{3x-2y}{7}\right)(1-2i)$$

So given vectors $2+3i$ & $1-2i$ generate C over R.

Not imp

Q17: Let S and T be subsets of a vector space V. Show that

(i) $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Give an example to show that equality need not hold in either case.

Sol:-

(i) Let $S = \{u_1, u_2, \ldots \ldots, u_n\}$

& $T = \{v_1, v_2, \ldots\ldots, v_t\}$

$\Rightarrow S \cup T = \{u_1, u_2, \ldots\ldots, u_r, v_1, v_2, \ldots\ldots, v_t\}$

we want to show $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

Let $v \in \langle S \rangle \cup \langle T \rangle$

$\Rightarrow v \in \langle S \rangle$ or $v \in \langle T \rangle$

If $v \in \langle S \rangle$ then v is a linear Combination of vectors of S

i.e., $v = K_1 u_1 + K_2 u_2 + \ldots\ldots + K_r u_r$

we can also write v as

$v = K_1 u_1 + K_2 u_2 + \ldots\ldots + K_r u_r + 0 v_1 + 0 v_2 + \ldots\ldots + 0 v_t$

which shows that v is a linear Combination of vectors of $S \cup T$

$\Rightarrow v \in \langle S \cup T \rangle$

So $v \in \langle S \rangle \Rightarrow v \in \langle S \cup T \rangle$

Similarly $v \in \langle T \rangle \Rightarrow v \in \langle S \cup T \rangle$

Hence $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

Now we give an example to show that equality does not hold in above result.

<u>Example</u> In R^2, Let $S = \{(1,0)\}$ & $T = \{(0,1)\}$ then

$\langle S \rangle = \{K(1,0) : K \in R\} = \{(K,0) : K \in R\}$ = x-axis

& $\langle T \rangle = \{l(0,1) : l \in R\} = \{(0,l) : l \in R\}$ = y-axis

So $\langle S \rangle \cup \langle T \rangle = \{(K,0) : K \in R\} \cup \{(0,l) : l \in R\} \neq R^2$

Now $S \cup T = \{(1,0), (0,1)\}$

Any vector $(x,y) \in R^2$ is a linear Combination of $(1,0)$ & $(0,1)$.

because $(x,y) = x(1,0) + y(0,1)$

Therefore $\langle S \cup T \rangle = R^2$

This shows that $\langle S \rangle \cup \langle T \rangle \neq \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Sol:.

First we prove that if $S \subset T$ then $\langle S \rangle \subset \langle T \rangle$.

Let $S = \{v_1, v_2, \ldots, v_r\}$

& $T = \{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$

then obviously $S \subset T$

we will show that $\langle S \rangle \subset \langle T \rangle$

Let $u \in \langle S \rangle$

then u is a linear combination of vectors of S

i.e., $u = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r$

Now u can also be written as

$u = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r + 0 u_{r+1} + \cdots + 0 u_n$

which shows that u is a linear combination of vectors of T. Hence $u \in \langle T \rangle$

So $\langle S \rangle \subset \langle T \rangle$

Hence $S \subset T \Rightarrow \langle S \rangle \subset \langle T \rangle$

Now as $S \cap T \subset S$ & $S \cap T \subset T$

$\Rightarrow \langle S \cap T \rangle \subset \langle S \rangle$ & $\langle S \cap T \rangle \subset \langle T \rangle$

$\Rightarrow \langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Now we will give an example to show that equality does not hold in above result

Example

In R^2, let $S = \{(0,0),(0,1)\}$ & $T = \{(0,0),(0,3)\}$

then $S \cap T = \{(0,0)\}$

So $\langle S \cap T \rangle = \{k(0,0) : k \in R\}$

$= \{(0,0)\}$ ——— ①

Now $\langle S \rangle = \{a(0,0)+b(0,1) : a,b \in R\}$

$= \{(0,b) : b \in R\}$

& $\langle T \rangle = \{p(0,0)+q(0,3) : p,q \in R\}$

$= \{(0,3q) : q \in R\}$

$= \{(0,c) : c \in R\}$

Now $\langle S \rangle \cap \langle T \rangle = \{(0,b) : b \in R\} \cap \{(0,c) : c \in R\}$

$= \{(0,y) : y \in R\}$ ——— ②

from ① & ②, we see

$\langle S \cap T \rangle \neq \langle S \rangle \cap \langle T \rangle$

Suppose $x+y \in X$

then $y = (x+y) - x \in X$ ($\because X$ is a vector space)

which is contradiction

Similarly if $x+y \in Y$

then $x = (x+y) - y \in Y$ ($\because Y$ is a vector space)

which is again contradiction

Hence our supposition is wrong

So either $X \subset Y$ or $Y \subset X$

Conversely

Let $X \subset Y$ or $Y \subset X$

$\Rightarrow X \cup Y = Y$ or $X \cup Y = X$

Since X & Y are subspaces of V

Hence $X \cup Y$ is also a subspace of V.

Q6: Which of the following are subspaces of $\mathbb{R}^3$?

(i) $W = \{(x,y,z) : x+y+z = 0\}$

Sol:

Here $W = \{(x,y,z) : x+y+z = 0\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1+y_1+z_1 = 0$ & $x_2+y_2+z_2 = 0$

Now let $a, b \in \mathbb{R}$ then