Exercise 6.2

Q1: Determine whether the following vectors in R^4 are linearly independent or linearly dependent.

(i) $(1,3,-1,4),(3,8,-5,7),(2,9,4,23)$

Sol.

Given vectors are $(1,3,-1,4),(3,8,-5,7),(2,9,4,23)$

Let $a(1,3,-1,4)+b(3,8,-5,7)+c(2,9,4,23)=0 \quad a,b,c \in F$

or $(a,3a,-a,4a)+(3b,8b,-5b,7b)+(2c,9c,4c,23c)=0$

$(a+3b+2c,\ 3a+8b+9c,\ -a-5b+4c,\ 4a+7b+23c)=0$

$\Rightarrow \quad a+3b+2c=0 \quad \text{——} (1)$

$3a+8b+9c=0 \quad \text{——} (2)$

$-a-5b+4c=0 \quad \text{——} (3)$

$4a+7b+23c=0 \quad \text{——} (4)$

from (1) & (2)

$$\frac{a}{27-16}=\frac{-b}{9-6}=\frac{c}{8-9}$$

$$\frac{a}{11}=\frac{b}{-3}=\frac{c}{-1}=k$$

$\Rightarrow a=11k$

$b=-3k$

$c=-k$

Putting these values in (3) & (4), we see eq. (3) & (4) are satisfied.

Hence given vectors in R^4 are linearly dependent.

(ii) $(1,-2,4,1),(2,1,0,-3),(1,-6,1,4)$

Sol:- Given vectors are $(1,-2,4,1),(2,1,0,-3),(1,-6,1,4)$

Let $a(1,-2,4,1)+b(2,1,0,-3)+c(1,-6,1,4)=0 \quad a,b,c\in F$

or $(a,-2a,4a,a)+(2b,b,0,-3b)+(c,-6c,c,4c)=0$

or $(a+2b+c,-2a+b-6c,4a+c,a-3b+4c)=0$

$\Rightarrow \quad a+2b+c=0$ —— ①

$-2a+b-6c=0$ —— ②

$4a+c=0$ —— ③

$a-3b+4c=0$ —— ④

from ① + ②

$$\frac{a}{-12-1}=\frac{-b}{-6+2}=\frac{c}{1+4}$$

$$\frac{a}{-13}=\frac{b}{4}=\frac{c}{5}=K$$

$\Rightarrow \quad a=-13K$

$b=4K$

$c=5K$

Putting these values in ③ + ④, we see that eqs. are not satisfied. They are satisfied only when

$K=0$

$\Rightarrow \quad a=b=c=0$

Hence given vectors in R^4 are linearly independent.

Q2: Let $V = P_3(x)$ be the vector space of all polynomials of degree ≤ 3 over R together with the zero polynomial. Determine whether $u, v, w \in V$ are linearly dependent or linearly independent.

(i) $u = x^3-4x^2+2x+3$, $v = x^3+2x^2+4x-1$, $w = 2x^3-x^2-3x+3$

Sol. Given vectors are

$$u = x^3-4x^2+2x+3$$

$$v = x^3+2x^2+4x-1$$

$$w = 2x^3-x^2-3x+3$$

Let $au+bv+cw = 0$ where $a, b, c \in F$.

or $a(x^3-4x^2+2x+3)+b(x^3+2x^2+4x-1)+c(2x^3-x^2-3x+3) = 0$

$(a+b+2c)x^3+(-4a+2b-c)x^2+(2a+4b-3c)x+(3a-b+3c) = 0$

$$\Rightarrow \quad a+b+2c = 0 \quad \text{——— (1)}$$

$$-4a+2b-c = 0 \quad \text{——— (2)}$$

$$2a+4b-3c = 0 \quad \text{——— (3)}$$

$$3a-b+3c = 0 \quad \text{——— (4)}$$

from (1) & (2)

$$\frac{a}{-1-4} = \frac{-b}{-1+8} = \frac{c}{2+4}$$

$$\frac{a}{-5} = \frac{b}{-7} = \frac{c}{6} = K$$

$$\Rightarrow \quad a = -5K$$

$$b = -7K$$

$$c = 6K$$

Putting these values in (3) & (4) we see equations are not satisfied. They are satisfied only when $K = 0$

$\Rightarrow a = b = c = 0$

Hence vectors u, v, w are linearly independent

(iii) $u = x^3 - 3x^2 - 2x + 3$, $v = x^3 - 4x^2 - 3x + 4$, $w = 2x^3 - 7x^2 - 7x + 9$

Sol. Given vectors are

$u = x^3 - 3x^2 - 2x + 3$

$v = x^3 - 4x^2 - 3x + 4$

$w = 2x^3 - 7x^2 - 7x + 9$

Let $au + bv + cw = 0$ where $a, b, c \in F$

or $a(x^3 - 3x^2 - 2x + 3) + b(x^3 - 4x^2 - 3x + 4) + c(2x^3 - 7x^2 - 7x + 9) = 0$

$(a+b+2c)x^3 + (-3a-4b-7c)x^2 + (-2a-3b-7c)x + (3a+4b+9c) = 0$

$\Rightarrow a + b + 2c = 0$ ——— (1)

$-3a - 4b - 7c = 0$ ——— (2)

$-2a - 3b - 7c = 0$ ——— (3)

$3a + 4b + 9c = 0$ ——— (4)

From (1) & (2)

$$\frac{a}{-7+8} = \frac{-b}{-7+6} = \frac{c}{-4+3}$$

$$\frac{a}{1} = \frac{b}{1} = \frac{c}{-1} = K$$

$\Rightarrow a = K$

$b = K$

$c = -K$

Putting these values in ③ & ④, we see eqs. are not satisfied. They are satisfied only when $K=0$

$\Rightarrow a=b=c=0$

Hence vectors u, v, w are linearly independent

Q3: Show that the vectors $(1-i, i)$ & $(2, -1+i)$ in C^2 are linearly dependent over C but linearly independent over R.

Sol:

Given vectors are $(1-i, i)$ & $(2, -1+i)$

Let $a(1-i, i) + b(2, -1+i) = 0$ where $a, b \in F$

or $(a(1-i), ai) + (2b, b(-1+i)) = 0$

$(a(1-i)+2b, ai+b(-1+i)) = 0$

$\Rightarrow \quad a(1-i) + 2b = 0$ ——— ①

$ai + b(-1+i) = 0$ ——— ②

These eqs. are satisfied in R only when $a=b=0$

Hence given vectors are linearly independent over R.

Now we find the values of a & b from the set C which satisfy eq. ① & ②.

from ① $a(1-i) = -2b$

$$\frac{a}{b} = \frac{-2}{1-i}$$

$$= \frac{-2}{1-i} \times \frac{1+i}{1+i}$$

$$\frac{a}{b} = \frac{-2(1+i)}{1-i^2}$$
$$= \frac{-2(1+i)}{1+1}$$
$$= \frac{-2(1+i)}{2}$$
$$\frac{a}{b} = -(1+i)$$
or $\frac{a}{1+i} = \frac{b}{-1} = K$

$\Rightarrow$ $a = (1+i)K$

$b = -K$

Putting these values in eq-②, we see eq ② is satisfied.

Hence given vectors are linearly dependent over C

Q4: Show that the vectors $(3+\sqrt{2}, 1+\sqrt{2})$ & $(7, 1+2\sqrt{2})$ in R^2 are linearly dependent over R but linearly independent over Q.

Sol:-

Given vectors are $(3+\sqrt{2}, 1+\sqrt{2})$ & $(7, 1+2\sqrt{2})$

Let $a(3+\sqrt{2}, 1+\sqrt{2}) + b(7, 1+2\sqrt{2}) = 0$

or $\left(a(3+\sqrt{2}), a(1+\sqrt{2})\right) + \left(7b, b(1+2\sqrt{2})\right) = 0$

or $\left(a(3+\sqrt{2})+7b, a(1+\sqrt{2})+b(1+2\sqrt{2})\right) = 0$

$\Rightarrow$ $(3+\sqrt{2})a + 7b = 0$ ——— ①

& $(1+\sqrt{2})a + (1+2\sqrt{2})b = 0$ ——— ②

These eqs. are satisfied in Q only when $a = b = 0$

Hence given vectors are linearly independent over Q.

Now we will find values of a & b in R which satisfy the equations ① & ②.

From ① $(3+\sqrt{2})a = -7b$

$$\frac{a}{b} = \frac{-7}{3+\sqrt{2}}$$

$$\frac{a}{7} = \frac{-b}{3+\sqrt{2}} = K$$

$$\Rightarrow a = 7K$$

$$b = -(3+\sqrt{2})K$$

Putting these values in eq. ②, we see eq. ② is satisfied.

Hence given vectors are linearly dependent over R.

Q5: Suppose that u, v, w are linearly independent vectors. Prove that

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(i) $u+v-2w$, $u-v-w$, $u+w$ are linearly independent.

Sol:.

Given vectors are $u+v-2w$, $u-v-w$ & $u+w$

Let $a(u+v-2w) + b(u-v-w) + c(u+w) = 0$ where $a, b, c \in F$

or $(a+b+c)u + (a-b)v + (-2a-b+c)w = 0$

As u, v, w are linearly independent

So $a+b+c = 0$ ——— ①

$a-b = 0$ ——— ②

$-2a-b+c = 0$ ——— ③

from ① + ③

$$\frac{a}{1+1} = \frac{-b}{1+2} = \frac{c}{-1+2}$$

$$\frac{a}{2} = \frac{b}{-3} = \frac{c}{1} = k$$

$\Rightarrow$ $a = 2k$

$b = -3k$

$c = k$

Putting these values in ②, we see that eq ② is not satisfied. It is satisfied only when $k = 0$

$\Rightarrow$ $a = b = c = 0$

Hence given vectors are linearly independent.

(ii) $u+v-3w$, $u+3v-w$, $u+w$ are linearly independent;

Sol:-

Given vectors are $u+v-3w$, $u+3v-w$, $u+w$

Let $a(u+v-3w)+b(u+3v-w)+c(u+w)=0$ where $a, b, c \in F$

or $(a+b+c)u+(a+3b)v+(-3a-b+c)w = 0$

As u, v + w are linearly independent.

$\Rightarrow$ $a+b+c = 0$ —— ①

$a+3b = 0$ —— ②

$-3a-b+c = 0$ —— ③

from ① + ③

$$\frac{a}{1+1} = \frac{-b}{1+3} = \frac{c}{-1+3}$$

$$\frac{a}{2} = \frac{b}{-4} = \frac{c}{2} = K$$

$\Rightarrow a = 2K$

$b = -4K$

$c = 2K$

Putting these values in eq ①, we see eq ② is not satisfied It is satisfied only when $K = 0$

$\Rightarrow a = b = c = 0$

Hence given vectors are linearly independent.

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Q6: Determine K so that the vectors $(1,-1,K-1),(2,K,-4),(0,2+K,-8)$ in R^3 are linearly dependent.

Sol:.

Suppose that given vectors are linearly dependent. So one of them must be a linear Combination of the other two vectors.

Let $(1,-1,K-1) = a(2,K,-4)+b(0,2+K,-8)$ where $a,b \in F$

$= (2a, aK, -4a)+(0, b(2+K), -8b)$

$(1,-1,K-1) = (2a, aK + b(2+K), -4a-8b)$

$\Rightarrow 2a = 1$ ——— ①

$aK + b(2+K) = -1$ ——— ②

$-4a - 8b = K-1$ ——— ③

from ① $a = \frac{1}{2}$

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Put in ② & ③

② $\Rightarrow \frac{k}{2} + b(2+k) = -1$ ——— ④

③ $\Rightarrow -4\left(\frac{1}{2}\right) - 8b = k-1$

$$-2 - 8b = k-1$$

$$-8b = k-1+2$$

$$-8b = k+1$$

$$b = -\frac{k+1}{8}$$

Put this value in ④

$$\frac{k}{2} + (2+k)\left(-\frac{k+1}{8}\right) = -1$$

$$4k - (2+k)(k+1) = -8$$

$$4k - 2k - 2 - k^2 - k + 8 = 0$$

$$-k^2 + k + 6 = 0$$

$$k^2 - k - 6 = 0$$

$$k^2 - 3k + 2k - 6 = 0$$

$$k(k-3) + 2(k-3) = 0$$

$$(k-3)(k+2) = 0$$

$$\Rightarrow k = 3, -2$$

Hence for $k = 3, -2$, given vectors are linearly dependent.

Q7: Using the technique of casting out vectors which are linear combination of others, find a linearly independent subset of the given set spanning the same subspace

(i) $\{(1,-3,1), (2,1,-4), (-2,6,-2), (-1,10,-7)\}$ in R^3.

Sol:. Given set is

$$\{(1,-3,1),(2,1,-4),(-2,6,-2),(-1,10,-7)\}$$

We see that

$$(-2,6,-2) = -2(1,-3,1)$$

So we cast out $(-2,6,-2)$ & obtain the subset

$$\{(1,-3,1),(2,1,-4),(-1,10,-7)\}$$

Suppose $(-1,10,-7) = a(1,-3,1)+b(2,1,-4)$ for $a,b \in R$

$$= (a,-3a,a)+(2b,b,-4b)$$

$$(-1,10,-7) = (a+2b, -3a+b, a-4b)$$

$$\Rightarrow \quad a+2b = -1 \quad \text{——— } ①$$

$$-3a+b = 10 \quad \text{——— } ②$$

$$a-4b = -7 \quad \text{——— } ③$$

Multiplying ① by 2 & adding in ③

$$2a+\cancel{4b} = -2 \quad \text{——— } ④$$

$$a-\cancel{4b} = -7 \quad \text{——— } ③$$

$$3a = -9$$

$$\boxed{a=-3}$$

Put in ③

$$-3-4b = -7$$

$$-4b = -7+3$$

$$-4b = -4$$

$$\boxed{b=1}$$

So $(-1,10,-7) = -3(1,-3,1) + 1(2,1,-4)$

Hence $(-1,10,-7)$ is a linear Combination of $(1,-3,1)$ & $(2,1,-4)$.

we cast out $(-1,10,-7)$ & obtain a subset

$A = \{(1,-3,1),(2,1,-4)\}$

Since $(2,1,-4)$ is not a scalar multiple of $(1,-3,1)$

So the set $A = \{(1,-3,1),(2,1,-4)\}$ is required linearly independent set which spans the same subspace as the given set of four vectors.

(ii) $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$ in the space of all functions from R to R.

Sol:-

Given set is $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$

As $\cos 2x = \cos^2 x - \sin^2 x$

or $\cos 2x = 1.\cos^2 x + (-1)\sin^2 x$

So $\cos 2x$ is a linear Combination of $\cos^2 x$ & $\sin^2 x$.

we cast out $\cos 2x$ & obtain the subset

$\{1, \sin^2 x, \cos^2 x\}$

Also $\cos^2 x = 1 - \sin^2 x$

or $\cos^2 x = 1.1 + (-1)\sin^2 x$

So $\cos^2 x$ is a linear Combination of 1 & $\sin^2 x$.

We cast out $\cos^2 x$ & obtain the subset $\{1, \sin^2 x\}$

Since none of 1 & $\sin^2 x$ is a scalar multiple of other, so $\{1, \sin^2 x\}$ is required linearly independent set which spans the same subspace as the given set of four vectors.

(iii) $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$ in the space $P_2(x)$ of polynomials.

Sol:-

Given set is $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$

We see that

$$3x-4 = \frac{3}{4}(4x+3) - \frac{25}{4}(1)$$

So $3x-4$ is a linear combination of $4x+3$ & 1.

We cast out $3x-4$ & obtain the subset

$$\{1, 4x+3, x^2+2, x-x^2\}$$

Now

Let $x-x^2 = a(1) + b(4x+3) + c(x^2+2)$

$x-x^2 = (a+3b+2c) + 4bx + cx^2$

$\Rightarrow$ $a+3b+2c = 0$ ——— ①

$4b = 1$ ——— ②

$c = -1$ ——— ③

from ③ $\boxed{c=-1}$

from ② $\boxed{b = \frac{1}{4}}$

① $\Rightarrow$ $a + \frac{3}{4} - 2 = 0$

$a - \frac{5}{4} = 0$

$\boxed{a = 5/4}$

so $x - x^2 = \frac{5}{4}(1) + \frac{1}{4}(4x+3) - 1(x^2+2)$

Hence $x - x^2$ is a linear Combination of $1, 4x+3$ & x^2+2.

we cast out $x - x^2$ & obtain the subset

$A = \{1, 4x+3, x^2+2\}$

we check whether A is linearly independent or not.

let $a(1) + b(4x+3) + c(x^2+2) = 0$

$(a + 3b + 2c) + 4bx + cx^2 = 0$

$\Rightarrow$ $a + 3b + 2c = 0$ ——— ①

$4b = 0$ ——— ②

$c = 0$ ——— ③

from ③ $\boxed{c = 0}$

from ② $\boxed{b = 0}$

Put in ①

$a + 0 + 0 = 0$

$\boxed{a = 0}$

Hence the set $A = \{1, 4x+3, x^2+2\}$ is linearly independent set which spans the same subspace as the given set of five vectors.

Q8: Verify that the polynomials $2-x^2$, x^3-x, $2-3x^2$ & $3-x^3$ form a basis for $P_3(x)$. Express each of (i) $1+x$ & (ii) $x+x^2$ as a linear combination of these basis vectors.

Sol:-

We want to show that $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$ forms a basis for $P_3(x)$.

First we prove that this set is linearly independent.

let $a(2-x^2)+b(x^3-x)+c(2-3x^2)+d(3-x^3)=0$; $a,b,c,d \in F$

$(2a+2c+3d)-bx+(-a-3c)x^2+(b-d)x^3=0$

$\Rightarrow$ $2a+2c+3d=0$ ——— (1)

$-b=0$ ——— (2)

$-a-3c=0$ ——— (3)

$b-d=0$ ——— (4)

from (2) $\boxed{b=0}$

from (4) $\boxed{d=0}$

from (3) $a=-3c$

Put in (1)

$2(-3c)+2c+3(0)=0$

$-4c = 0$

$\boxed{c = 0}$

Put in 3

$-a - 3(0) = 0$

$\boxed{a = 0}$

Hence the given set of polynomials is linearly independent.

As dimension of $P_3(x)$ is 4

& no. of vectors in set $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$ is 4.

So the given set forms a basis for $P_3(x)$.

(i) Now we express $1+x$ as a linear Combination of given vectors.

let $1+x = a(2-x^2) + b(x^3-x) + c(2-3x^2) + d(3-x^3)$

$1+x = (2a+2c+3d) - bx + (-a-3c)x^2 + (b-d)x^3$; $a,b,c,d \in R$

$\Rightarrow 2a + 2c + 3d = 1$ ——— 1

$-b = 1$ ——— 2

$-a - 3c = 0$ ——— 3

$b - d = 0$ ——— 4

from 2 $\boxed{b = -1}$

Put in 4

$-1 - d = 0$

$\boxed{d = -1}$

from ③ $a = -3c$

Put in ①

$2(-3c) + 2c + 3(-1) = 1$

$-6c + 2c - 3 = 1$

$-4c = 1+3$

$-4c = 4$

$\boxed{c = -1}$

Put in ③

$-a - 3(-1) = 0$

$-a + 3 = 0$

$\boxed{a = 3}$

So $1 + x = 3(2-x^3) - (x^3 - x) - (2 - 3x^2) - (3 - x^3)$

(ii) Now we express $x + x^2$ as a linear combination of given vectors.

Let $x + x^2 = a(2-x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3)$; $a, b, c, d \in R$

$x + x^2 = (2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3$

$\Rightarrow$ $2a + 2c + 3d = 0$ ———— ①

$-b = 1$ ———— ②

$-a - 3c = 1$ ———— ③

$b - d = 0$ ———— ④

from ② $\boxed{b = -1}$

from ④ $-1 - d = 0$

$\boxed{d = -1}$

from ③ $\quad a = -1 - 3c$

Put in ①

$2(-1-3c) + 2c + 3(-1) = 0$

$-2 - 6c + 2c - 3 = 0$

$-4c - 5 = 0$

$-4c = 5$

$\boxed{c = -5/4}$

Put in ③

$-a - 3(-5/4) = 1$

$-a + \frac{15}{4} = 1$

$a = \frac{15}{4} - 1$

$\boxed{a = \frac{11}{4}}$

So $x + x^2 = \frac{11}{4}(2 - x^2) - (x^3 - x) - \frac{5}{4}(2 - 3x^2) - (3 - x^3)$

Q4: Determine whether or not the given set of vectors is a basis for R^2.

(i) $\{(1,1), (3,1)\}$

Sol:-

Given set is $\{(1,1),(3,1)\}$

First we will check their linear independency.

Let $a(1,1) + b(3,1) = 0 \quad$ where $a, b \in R$

or $(a+3b, a+b) = 0$

$\Rightarrow \quad a+3b = 0$ ——— (1)

$a+b = 0$ ——— (2)

Subt (2) from (1)

$2b = 0$

$\boxed{b=0}$

Put in (1)

$a+0 = 0$

$\boxed{a=0}$

Hence given set of vectors is linearly independent

As dimension of R^2 is 2.

& no. of linearly independent vectors is also 2.

So the set $\{(1,1),(3,1)\}$ forms a basis for R^2.

(ii) $\{(2,1),(1,-1)\}$

Sol:

Given set is $\{(2,1),(1,-1)\}$

First we will check their linear independency.

let $a(2,1)+b(1,-1) = 0$ where $a, b \in R$

$(2a, a) + (b, -b) = 0$

$(2a+b, a-b) = 0$

$\Rightarrow \quad 2a+b = 0$ ——— (1)

$a-b = 0$ ——— (2)

Adding ① & ②

$3a = 0$

$\boxed{a = 0}$

Put in ②

$0 - b = 0$

$\boxed{b = 0}$

Hence the given set $\{(2,1),(1,-1)\}$ is linearly independent.
Since dimension of R^2 is 2.
& no. of linearly independent vectors is also 2
So $\{(2,1),(1,-1)\}$ forms a basis for R^2.

Q10: Determine whether or not the given set of vectors is a basis for R^3.

(i) $\{(1,2,-1),(0,3,1),(1,-5,3)\}$

Sol:-

Given set is $\{(1,2,-1),(0,3,1),(1,-5,3)\}$.

First we will check their linear independency.

Let $a(1,2,-1) + b(0,3,1) + c(1,-5,3) = 0$; $a,b,c \in R$

$(a,2a,-a) + (0,3b,b) + (c,-5c,3c) = 0$

$(a+c,\ 2a+3b-5c,\ -a+b+3c) = 0$

$\Rightarrow a + c = 0$ ——— ①

$2a + 3b - 5c = 0$ ——— ②

$-a + b + 3c = 0$ ——— ③

from ② & ③

$$\frac{a}{9+5} = \frac{-b}{6-5} = \frac{c}{2+3}$$

$$\frac{a}{14} = \frac{b}{-1} = \frac{c}{5} = K$$

$\Rightarrow a = 14K$

$b = -K$

$c = 5K$

Putting these values in ①, we see eq. ① is not satisfied. It is satisfied only when $K = 0$.

$\Rightarrow a = b = c = 0$

Hence the given set $\{(1,2,-1), (0,3,1), (1,-5,3)\}$ is linearly independent.

Since dimension of R^3 is 3

& no. of linearly independent vectors is also 3.

So the set $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ forms a basis for R^3.

(ii) $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

Sol:.

Given set is $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

First we will check their linear independency

Let $a(2,4,-3) + b(0,1,1) + c(0,1,-1) = 0$ where $a,b,c \in R$

$(2a, 4a, -3a) + (0, b, b) + (0, c, -c) = 0$

$(2a, 4a+b+c, -3a+b-c) = 0$

$$4a+b+c=0 \quad \text{——— (2)}$$

$$-3a+b-c=0 \quad \text{——— (3)}$$

from (1) $\boxed{a=0}$

Put in (2) & (3)

$$b+c=0 \quad \text{——— (2)}$$

$$b-c=0 \quad \text{——— (3)}$$

Adding (2) & (3)

$$2b=0$$

$$\boxed{b=0}$$

Put in (2)

$$0+c=0$$

$$\boxed{c=0}$$

Hence given set $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ is linearly independent.

Since dimension of $R^3 = 3$

& no. of linearly independent vectors is also 3.

So the set $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ forms a basis for R^3.

Q11: Let V be the real vector space of all functions defined on R into R. Determine whether the given vectors are linearly independent or linearly dependent in V

(i) $x, \cos x$

Sol.

Given vectors are $x, \cos x$

Suppose $ax + b\cos x = 0$ —— ① where $a, b \in R$ & $x \in R$

Put $x = 0$ in ①

$a(0) + b\cos 0 = 0$

$0 + b = 0$

$\boxed{b = 0}$

Put $x = \pi/2$ in ①

$a\left(\frac{\pi}{2}\right) + b\cos \pi/2 = 0$

$a(\pi/2) + b(0) = 0$

$a(\pi/2) = 0$

$\boxed{a = 0}$

Hence given vectors are linearly independent.

(ii) $\sin^2 x, \cos^2 x, \cos 2x$

Sol.

Given vectors are $\sin^2 x, \cos^2 x, \cos 2x$

Let $a\sin^2 x + b\cos^2 x + c\cos 2x = 0$ —— ① $a, b, c \in R$, $x \in R$

Put $x = 0$ in ①

$a\sin^2(0) + b\cos^2(0) + c\cos 0 = 0$

$0 + b + c = 0$

$b + c = 0$ ——— (2)

Put $x = \frac{\pi}{2}$ in (1).

$a\sin^2\frac{\pi}{2} + b\cos^2\frac{\pi}{2} + c\cos\pi = 0$

$a + 0 - c = 0$

$a - c = 0$ ——— (3)

Put $x = \frac{\pi}{4}$ in (1)

$a\sin^2(\pi/4) + b\cos^2\pi/4 + c\cos\pi/2 = 0$

$a\left(\frac{1}{2}\right) + b\left(\frac{1}{2}\right) + c(0) = 0$

$\frac{a}{2} + \frac{b}{2} = 0$

$a + b = 0$ ——— (4)

(2) $\Rightarrow$ $b = -c$

(3) $\Rightarrow$ $a = c$

So nonzero solution of above eqs. is

$a = c$

$b = -c$

$c = c$ or $(c, -c, c)$; $c \in R$

So given vectors are linearly dependent

(iii) $\sin x, \cos x, \sinh x, \cosh x$

Sol:-

Given vectors are $\sin x, \cos x, \sinh x, \cosh x$

let $a\sin x + b\cos x + c\sinh x + d\cosh x = 0$ ——— (A)

Put $x = 0$ in (A)

$a\sin 0 + b\cos 0 + c\sinh 0 + d\cosh 0 = 0$

$0 + b + 0 + d = 0$

$b + d = 0$ ——— (1)

Diff. (A) w.r.t. x

$a\cos x - b\sin x + c\cosh x + d\sinh x = 0$ ——— (B)

Put $x = 0$

$a\cos 0 - b\sin 0 + c\cosh 0 + d\sinh 0 = 0$

$a - 0 + c + 0 = 0$

$a + c = 0$ ——— (2)

Diff. (B) w.r.t. x

$-a\sin x - b\cos x + c\sinh x + d\cosh x = 0$ ——— (C)

Put $x = 0$

$-a\sin 0 - b\cos 0 + c\sinh 0 + d\cosh 0 = 0$

$-0 - b + 0 + d = 0$

$-b + d = 0$ ——— (3)

Diff. (C) w.r.t. x

$-a\cos x + b\sin x + c\cosh x + d\sinh x = 0$

Put $x = 0$

$-a\cos 0 + b\sin 0 + c\cosh 0 + d\sinh 0 = 0$

$-a + 0 + c + 0 = 0$

$-a + c = 0$ ——— (4)

Adding (1) & (3)

$\boxed{d=0}$

Put in (1)

$b+0=0$

$\boxed{b=0}$

Adding (2) & (4)

$2c=0$

$\boxed{c=0}$

Put in (4)

$-a+0=0$

$\boxed{a=0}$

Hence given vectors are linearly independent.

(iv) $\sin x$, $\sin x+\cos x$, $\sin x-\cos x$

Sol:-

Given vectors are $\sin x$, $\sin x+\cos x$, $\sin x-\cos x$

let $a\sin x+b(\sin x+\cos x)+c(\sin x-\cos x)=0$ $\quad a, b, c\in R$

or $(a+b+c)\sin x+(b-c)\cos x=0$ ———— (A)

Put $x=0$ in (A)

$(a+b+c)\sin 0+(b-c)\cos 0=0$

$b-c=0$ ———— (1)

Put $x=\frac{\pi}{2}$ in (A)

$$(a+b+c)\sin\pi/2 + (b-c)\cos\pi/2 = 0$$

$$a+b+c = 0 \quad \text{—— (2)}$$

Put $x = \pi$ in (A)

$$(a+b+c)\sin\pi + (b-c)\cos\pi = 0$$

$$(b-c)(-1) = 0$$

$$b-c = 0 \quad \text{—— (3)}$$

(3) $\Rightarrow$ $b = c$

Put in (2)

$$a+c+c = 0$$

$$a+2c = 0$$

$$\boxed{a = -2c}$$

So a nonzero solution of above eq. is $(-2c, c, c)$

Hence given, vectors are linearly dependent.

(v) e^{ax}, e^{bx}, e^{cx} ; a, b, c being distinct real no's.

Sol:.

Given vectors are e^{ax}, e^{bx}, e^{cx}

Let $\alpha e^{ax} + \beta e^{bx} + \gamma e^{cx} = 0$ —— (A) $\alpha, \beta, \gamma \in R$, $x \in R$

Put $x = 0, 1, -1$ in (A)

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha e^{a} + \beta e^{b} + \gamma e^{c} &= 0 \\ \alpha e^{-a} + \beta e^{-b} + \gamma e^{-c} &= 0 \end{aligned}\right]$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ e^{a} & e^{b} & e^{c} \\ e^{-a} & e^{-b} & e^{-c} \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ e^{a} & e^{b} & e^{c} \\ e^{-a} & e^{-b} & e^{-c} \end{vmatrix}$$

Expanding from R_1

$$|A| = 1(e^{b-c} - e^{-b+c}) - 1(e^{a-c} - e^{-a+c}) + 1(e^{a-b} - e^{-a+b})$$

$$= e^{b-c} - e^{c-b} - e^{a-c} + e^{c-a} + e^{a-b} - e^{b-a}$$

$\neq 0 \quad \because a, b, c$ are distinct real no's.

So matrix A is non singular

Hence rank $A = 3$

So trivial solution of system is

$$\alpha = \beta = \gamma = 0$$

Hence the vectors e^{ax}, e^{bx}, e^{cx} are linearly independent.

Q12: Determine a basis for each of the following subspaces of R^3:

(i) The plane $x - 2y + 5z = 0$

Sol:-

Given eq. of plane is $x - 2y + 5z = 0$

or $x = 2y - 5z$ where y, z are free variables

The above eq. in vector form can be written as

$$(x,y,z) = (2y-5z, y, z)$$

$$= (2y-5z, y+0, 0+z)$$

$$= (2y, y, 0) + (-5z, 0, z)$$

$$\Rightarrow (x,y,z) = y(2,1,0) + z(-5,0,1)$$

Thus given plane is spanned by vectors $(2,1,0)$ & $(-5,0,1)$. Since none of the vector is scalar multiple of other. So the set $\{(2,1,0),(-5,0,1)\}$ is linearly independent.

Hence $\{(2,1,0),(-5,0,1)\}$ forms a basis for given subspace of R^3.

(ii) The line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{6}$

Sol:-

Given eq. of line is

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{6} = t$$

$$\Rightarrow x = -2t$$

$$y = t$$

$$z = 6t$$

The above eq. in vector form can be written as

$$(x,y,z) = (-2t, t, 6t)$$

or $(x,y,z) = t(-2,1,6)$

Hence the given line is spanned by the vector $(-2,1,6)$. Also $(-2,1,6)$ being a nonzero single vector is linearly independent.

So $\{(-2,1,6)\}$ forms a basis for the given subspace of R^3.

(iii) All vectors of the form (a,b,c) where $3a-2b+c=0$

Sol:

Given eq. is $3a-2b+c=0$

or $c=-3a+2b$

The above eq. can be written in vector form as

$$(a,b,c) = (a,b,-3a+2b)$$
$$= (a+0,0+b,-3a+2b)$$
$$= (a,0,-3a)+(0,b,2b)$$

or $(a,b,c) = a(1,0,-3)+b(0,1,2)$

So given subspace is spanned by $(1,0,-3)$ & $(0,1,2)$.

Now since none of the vector is a scalar multiple of other.

So the vectors $(1,0,-3)$ & $(0,1,2)$ are linearly dependent. So the set $\{(1,0,-3),(0,1,2)\}$ forms a basis for the given subspace of R^3.

Q13: Find the dimension of the subspace $\{(x_1, x_2, x_3, x_4) : x_2 = x_3\}$ of R^4. Also determine a basis.

Sol:-

Let $W = \{(x_1, x_2, x_3, x_4) : x_2 = x_3\}$

Suppose (x_1, x_2, x_2, x_4) be a general vector of W

then we can write it as

$$(x_1, x_2, x_2, x_4) = (x_1, 0, 0, 0) + (0, x_2, x_2, 0) + (0, 0, 0, x_4)$$

$$= x_1(1, 0, 0, 0) + x_2(0, 1, 1, 0) + x_4(0, 0, 0, 1)$$

Which shows that $(x_1, x_2, x_2, x_4) \in W$ is a linear combination of vectors $(1, 0, 0, 0), (0, 1, 1, 0)$ & $(0, 0, 0, 1)$.

So the set $S = \{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ spans W.

Now we check the linear independency of set S.

Suppose that for scalars $a, b \in R$

$$a(1, 0, 0, 0) + b(0, 1, 1, 0) + c(0, 0, 0, 1) = (0, 0, 0, 0)$$

$$(a, 0, 0, 0) + (0, b, b, 0) + (0, 0, 0, c) = (0, 0, 0, 0)$$

$$(a, b, b, c) = (0, 0, 0, 0)$$

$$\Rightarrow \quad a = 0$$

$$b = 0$$

$$c = 0$$

Hence the set $S = \{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ is also linearly independent. Hence S is a basis for W

So dimension of W is 3.

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Q14: A subspace U of R^4 is spanned by the vectors $(1,0,2,3)$ & $(0,1,-1,2)$ & a subspace W is spanned by $(1,2,3,4), (-1,-1,5,0)$ & $(0,0,0,1)$. Find the dimensions of U & W.

Sol:

i) Let $S = \{(1,0,2,3), (0,1,-1,2)\}$.

Since these vectors span U

So we only check their linear independency.

Suppose that for scalars $a, b \in R$

$$a(1,0,2,3) + b(0,1,-1,2) = 0$$

$$(a,0,2a,3a) + (0,b,-b,2b) = 0$$

$$(a, b, 2a-b, 3a+2b) = (0,0,0,0)$$

$$\Rightarrow \quad a = 0$$

$$b = 0$$

$$2a - b = 0$$

$$3a + 2b = 0$$

$$\Rightarrow \boxed{a = 0}$$

$$\& \boxed{b = 0}$$

which shows that the set S is linearly independant

Hence $S = \{(1,0,2,3), (0,1,-1,2)\}$ forms a basis for U

So dimension of $U = 2$

Sol.

(iii) Let $B = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$

Since these vectors span W (given)

So we only check their linear independency

Suppose that for scalars $a, b \in R$

$$a(1,2,3,4) + b(-1,-1,5,0) + c(0,0,0,1) = 0$$

$$(a, 2a, 3a, 4a) + (-b,-b,5b,0) + (0,0,0,c) = 0$$

$$(a-b,\ 2a-b,\ 3a+5b,\ 4a+c) = (0,0,0,0)$$

$\Rightarrow$ $a - b = 0$ ——— (1)

$2a - b = 0$ ——— (2)

$3a + 5b = 0$ ——— (3)

$4a + c = 0$ ——— (4)

Subt. (2) from (1)

$-a = 0$

$\boxed{a = 0}$

Put in (1)

$0 - b = 0$

$\boxed{b = 0}$

Hence the set $B = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ forms a basis for W.

Hence dimension of $W = 3$

Q15: Suppose that U & W are distinct four dimensional subspaces of a vector space V of dimension six. Find the possible dimension of $U \cap W$.

Sol.

We are given that

$\dim U = 4$

$\dim W = 4$

& $\dim V = 6$

Since $U+W$ is a subspace of V

so $\dim(U+W) \leq \dim V = 6$

$\Rightarrow \dim(U+W) \leq 6$

Now as $U \subseteq U+W$ & $W \subseteq U+W$

So $\dim U \leq \dim(U+W) \leq 6$

or $4 \leq \dim(U+W) \leq 6$

Hence $\dim(U+W)$ is 4 or 5 or 6.

Since U & W are distinct, so they must be different by atleast one generator

So $\dim(U+W) > 4$

Hence $\dim(U+W) = 5$ or 6

As we know

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

or $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

(ii) If $\dim(U+W) = 5$

then $\dim(U\cap W) = 4+4-5$

$= 3$

(iii) If $\dim(U+W) = 6$

then $\dim(U\cap W) = 4+4-6$

$= 2$

Hence possible dimensions of $U\cap W$ are 2 or 3

Q16: Find a basis and dimension of the subspace W of R^4 spanned by

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(i) $(1,4,-1,3), (2,1,-3,-1), (0,2,1,-5)$

Sol:.

As the subspace W of R^4 is spanned by $(1,4,-1,3)$, $(2,1,-3,-1)$ & $(0,2,1,-5)$

So we only check their linear independency.

For this let for $a,b,c \in F$

$a(1,4,-1,3)+b(2,1,-3,-1)+c(0,2,1,-5) = 0$

$(a,4a,-a,3a)+(2b,b,-3b,-b)+(0,2c,c,-5c) = 0$

$(a+2b, 4a+b+2c, -a-3b+c, 3a-b-5c) = (0,0,0,0)$

$\Rightarrow \quad a+2b = 0$ ——— (1)

$4a+b+2c = 0$ ——— (2)

$-a-3b+c = 0$ ——— (3)

$3a-b-5c = 0$ ——— (4)

from ① & ②

$$\frac{a}{4-0} = \frac{-b}{2-0} = \frac{c}{1-8}$$

$$\frac{a}{4} = \frac{b}{-2} = \frac{c}{-7} = K$$

$\Rightarrow$ $a = 4K$

$b = -2K$

$c = -7K$

Put in ③ & ④, we see eq. are not satisfied and they are satisfied only when $K = 0$

i.e., when $a = b = c = 0$

which shows that given vectors are linearly independent.

So $\{(1,4,-1,3),(2,1,-3,-1),(0,2,1,-5)\}$ forms a basis for W.

Hence $\dim W = 3$

(ii) $(1,-4,-2,1),(1,-3,-1,2),(3,-8,-2,7)$

Sol:-

As the subspace W of R^4 is spanned by $(1,-4,-2,1)$, $(1,-3,-1,2)$ & $(3,-8,-2,7)$

So we only check their linear independency.

For this let for $a,b,c \in F$

$$a(1,-4,-2,1) + b(1,-3,-1,2) + c(3,-8,-2,7) = 0$$

$(a,-4a,-2a,a)+(b,-3b,-b,2b)+(3c,-8c,-2c,7c) = 0$

$(a+b+3c,\ -4a-3b-8c,\ -2a-b-2c,\ a+2b+7c) = (0,0,0,0)$

$\Rightarrow \quad a+b+3c = 0$ ——— (1)

$-4a-3b-8c = 0$ ——— (2)

$-2a-b-2c = 0$ ——— (3)

$a+2b+7c = 0$ ——— (4)

from (1) & (2)

$$\frac{a}{-8+9} = \frac{-b}{-8+12} = \frac{c}{-3+4}$$

$$\frac{a}{1} = \frac{b}{-4} = \frac{c}{1} = K$$

$\Rightarrow \quad a = K$

$b = -4K$

$c = K$

Put these values in (3) & (4), we see these eqs. are satisfied, so the given vectors are linearly dependent. Now we take only first two vectors $(1,-4,-2,1)$ & $(1,-3,-1,2)$ & check their linear independency.

Suppose for $a,b \in F$

$a(1,-4,-2,1)+b(1,-3,-1,2) = 0$

$(a,-4a,-2a,a)+(b,-3b,-b,2b) = 0$

$(a+b,\ -4a-3b,\ -2a-b,\ a+2b) = (0,0,0,0)$

$\Rightarrow \quad a+b = 0$ ——— (1)

$-4a-3b = 0$ ——— (2)

$-2a-b = 0$ ——— (3)

$a+2b = 0$ ——— (4)

Adding (1) & (3)

$-a = 0$

$\Rightarrow \boxed{a=0}$

Put in (1)

$0+b = 0$

$\boxed{b=0}$

Hence the vectors are linearly independent & so the set $\{(1,-4,-2,1),(1,-3,-1,2)\}$ forms a basis for W.

Hence $\dim W = 2$

11: Let U & W be 2-dimensional subspaces of R^3. how that $U \cap W \neq \{0\}$

l:

iven $\left.\begin{array}{l}\dim U = 2 \\ \& \ \dim W = 2\end{array}\right]$

If $U = W$

en $\dim U = \dim W = 2$

So $U \cap W \neq \{0\}$

Hence we suppose that $U \neq W$

This means that U & W are not spanned by the same set.

So $\dim(U+W) > 2$

Since U & W are subspaces of R^3

Hence $\dim(U+W) \leq 3$

So $2 < \dim(U+W) \leq 3$

$\Rightarrow \dim(U+W) = 3$

As $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

So $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2+2-3$

$\dim(U \cap W) = 1$

This shows that $U \cap W$ Contains a nonzero element & so $U \cap W \neq \{0\}$.

Q18: If two or more rows (Columns) of an $n \times n$ matrix A are linearly dependent, then show that $\det A = 0$

Sol:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & & a_{in} \\ a_{j1} & a_{j2} & & a_{jn} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an $n \times n$ matrix whose ith & jth rows are linearly dependent, where

$P_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$

& $P_j = [a_{j1}, a_{j2}, \ldots, a_{jn}]$ are two row vectors.

Since P_i & P_j are linearly dependent

So there exist scalars α, β not both zero such that $\alpha P_i + \beta P_j = 0$

$$\Rightarrow \alpha P_i = -\beta P_j$$

or $$P_i = -\frac{\beta}{\alpha} P_j$$

where $\alpha \neq 0$ (say)

$$= -\frac{\beta}{\alpha}[a_{j1}, a_{j2}, \ldots, a_{jn}]$$

$$P_i = \left[-\frac{\beta}{\alpha}a_{j1}, -\frac{\beta}{\alpha}a_{j2}, \ldots, -\frac{\beta}{\alpha}a_{jn}\right]$$

So

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta}{\alpha}a_{j1} & -\frac{\beta}{\alpha}a_{j2} & \cdots & -\frac{\beta}{\alpha}a_{jn} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta}{\alpha}a_{j1} & -\frac{\beta}{\alpha}a_{j2} & \cdots & -\frac{\beta}{\alpha}a_{jn} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= -\frac{\beta}{\alpha}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

taking $-\frac{\beta}{\alpha}$ common from R_i

$|A| = 0 \quad \because$ two rows are identical

Similarly we can prove that if more than two rows are linearly dependant then $|A| = 0$

Also if two or more than two columns of A are linearly dependant then we can show that $|A| = 0$

or $\det A = 0$